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Mathematical Tables
and other
Aids to Computation

A Quarterly Journal edited on behalf of the
Committee on Mathematical Tables
and Other Aids to Computation
by

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III • Number 23 • July, 1948

Published by

THE NATIONAL RESEARCH COUNCIL

NATIONAL RESEARCH COUNCIL

DIVISION OF PHYSICAL SCIENCES

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Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor R. C. ARCHIBALD, Brown University, Providence 12, R. I.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

The IBM Pluggable Sequence Relay Calculator

1. Introduction

The first two machines of this type were built during the war for the Aberdeen Proving Ground, were delivered in December 1944 and were in operation during the last eight months of the war. In November 1945 they were returned to the factory for the provision of increased storage capacity and other improvements. At that time three additional machines of the improved type were built—one for the Naval Proving Ground at Dahlgren and two for the Watson Scientific Computing Laboratory at Columbia University. The latter are available to scientists engaged in basic research. The machines were designed and built in the Engineering Department of the IBM by C. D. LAKE, B. M. DURFEE, and D. R. PIATT.

The machine reads numbers from punched cards, performs a sequence of calculations on them by means of relay networks, and punches the results in cards. The calculating unit is the fastest relay unit that has been announced (November 1947); six-digit multiplications are performed in 0.15 second, which is about three times as fast as that of any other relay multiplying unit. The basic operations include addition, subtraction, multiplication, division, square root, and column shift. There are 36 storage and computing registers; mathematical tables with monotonic argument may be used. The reading and recording speed is greater than that of any other existing calculator. Cards are read and punched at the rate of 100 per minute, and it is possible to read four cards simultaneously and punch a fifth at this rate.

In comparison with the IBM Sequence Controlled Calculator at Harvard, this machine is limited in internal storage capacity, number of significant figures, and flexibility of sequencing; on the other hand, multiplying speed is about 20 times as great, and the use of plugboard facilities and punched cards permits parallel operation (as distinguished from sequence operation), with further gain in efficiency.

While the speed in performing the arithmetical operations is less than that of electronic machines, the over-all production on general computing is extremely high because of the many other factors involved. These factors include the use of automatic control panels for quick change from one problem to another; efficient card reading, punching, and comparing facilities; convenience of operation; and the advantages of having twin machines in the same room. For a great many scientific problems, the internal storage capacity of all existing machines is quickly exceeded, and the over-all speed is determined by the facilities for reading and recording rather than by the speed of computing; in such cases these calculators, which take full advantage of punched card storage, are unequalled in efficiency. A simple example of this type of problem is the manipulation of large harmonic series described in §6.

The scope of the machine may be seen by comparing it with the standard IBM Calculating Punch (type 602). The principal differences include

(a) higher operating speed (in general it is about 10 times as fast as the Calculating Punch; additions are performed at the rate of 40 per second, and multiplications at the rate of 6 per second),

(b) greater storage capacity (it has 36 storage and computing registers of 6 or 12 digits each, permitting more complicated sequences with one run of the cards),

(c) more flexible card reading and punching facilities (it has two separate card feeding units with four reading stations; it can punch 100 eighty-column cards per minute),

(d) additional functions (square root is included; one card feed can be used for a mathematical table; the machine performs many functions of the collator and reproducer),

(e) fewer significant figures (basic capacity of the multiplier is $6 \times 6 = 12$; $12 \times 12 = 24$ may be performed by sequencing).

2. Card Reading and Punching

The card reading unit is similar to a standard IBM Reproducing Punch, except that a full reading station has been inserted ahead of the punching station to replace the conventional x-brushes. Cards in feed A pass a set of x-brushes and two complete reading stations, "1" and "2"; cards in feed B pass reading station "3," the punching station, and reading station "4."

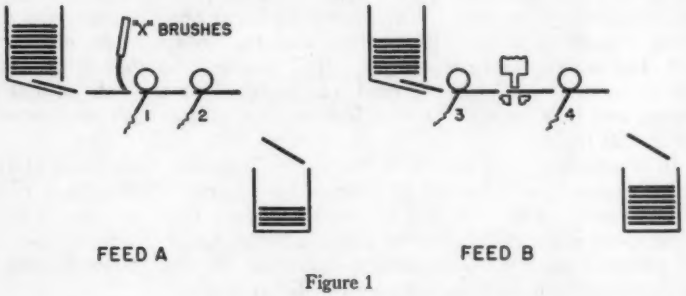

Figure 1

Data may be read from as many as 4 cards simultaneously and results punched on a fifth. Data may also be passed back from brush station "4" to the following card so that data from earlier cards in the series may also be available. The position of the x-brushes is such that they are reading "y" from one card at the same time that brushes "1" are reading the 9 of the preceding card. The two feeds may be operated continuously or intermittently, in synchronism or either one independently, as directed by the sequence control. The feed operates at the rate of 100 cards per minute (.6 second per card). A digit selector is included, and 24 dial switches provide for the storage in the machine of constant factors aggregating 24 digits.

There is a comparing unit of 80 columns, of the type used on the reproducing punch, which may be plugged to reading brushes or registers to stop the machine when the factors being compared are unequal. The comparing unit may be used to check the results of a previous run or to stop an itera-

tive process. It has pluggable outlets which allow it to be split into 8 independent units. Additional comparing and collating facilities will be listed in the next paragraph.

3. Relay Groups

The storage and computing units consist entirely of electromechanical relays arranged in 36 groups. These relay groups perform nine principal functions:

(a) receiving data from the cards
(b) receiving data from another relay group or from storage switches
(c) transmitting data to another group
(d) transmitting data to the punch magnets for recording on the cards or to the comparing magnets
(e) operating in conjunction with another group to form the sum or difference of the numbers in the groups
(f) operating in conjunction with other groups to form products
(g) operating in conjunction with other groups to form quotients
(h) operating in conjunction with other groups to form square roots
(i) operating in conjunction with another group to compare two numbers and indicate high-low-equal.

Each group is capable of performing two or more of these functions as shown in table 1 (the switch storage units are also shown). Some of the transfers of data take place between definite groups, and some can be performed at will over one of two main channels or "busses." Provision is made for connecting the two busses momentarily for transfers. The first column of the table lists the relay groups and the switch storage groups; the 12-digit groups are shown in two sections, 1–6 and 7–12. The next four columns indicate which groups can receive data from cards, from bus 1, from bus 2, and from other groups by direct connection. The following three columns indicate which groups may transmit data to cards, bus 1, and bus 2. In columns 9–12 are shown the groups which are used in the operations of cross addition (CA), multiplication (MP), division (Div), and square root (Sq Rt). A bracket in the columns headed Bus 1 and Bus 2 means that the result of a cross-adding operation is transmitted to the bus; a bracket in the direct transfer column indicates that two receiving groups (shown in the first column) are involved; a bracket in the CA column indicates the groups which may be cross-added. The final column indicates groups which are capable of transmitting complements or true figures as desired. Groups 19 and 20 and groups 24 and 25 can be used to compare two numbers for high-low-equal.

4. Basic Operations

The operations of all parts of the machine are synchronized with a rotating shaft in the card reading unit. This shaft turns at the rate of 100 revolutions per minute. When the card feeds are operated, they are moved in synchronism with this shaft so that a card is read or punched during one revolution of the shaft. During a revolution of the shaft a timing circuit produces 48 impulses (at 0.0125-second intervals) which may be used to operate relays, and a second timing circuit produces 14 impulses

which are used for reading and punching cards. Thus, 48 relay impulses are equivalent to 14 card pulses, or one card impulse is equivalent to $3\frac{3}{7}$ relay impulses. The cards are fed top or 12's edge first, and the 0 card impulse corresponds to the 13 relay impulse, as shown in figure 2. All relays are picked up and dropped out by the relay pulses, and all card reading, punching, and comparing are controlled by the card pulses.

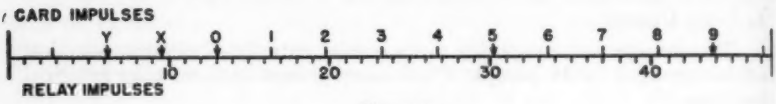

Figure 2

Table 1

Group	Receive from				Transmit to			CA	MP	Div	Sq Rt	Inv Read Out
	Cards	Bus 1	Bus 2	Direct Transfer	Cards	Bus 1	Bus 2					
MC	√		√				√		√			√
MP	√	√				√			√			√
1	√						√	√	√			
2	√							√	√			
3	√					√		√	√			
4	√							√	√			
5	√						√	√	√			
6	√							√	√			
7									√			
8									√			
9	√	√		11+12 (7–12)		√	√	√		√		
10	√		√	13			√	√				
11(1–6)	√	√				√		√		√	√	
(7–12)	√		√	9+10		√ (1–6)		√		√	√	
12(1–6)	√	√		16			√ (7–12)	√		√	√	
(7–12)	√		√					√		√	√	
13	√		√	9+10	√		√			√		√
14			√				√					√
15			√				√					√

Table 1 (cont.)

Group	Receive from				Transmit to			CA	MP	Div	Sq Rt	Inv Read Out
	Cards	Bus		Direct Transfer	Cards	Bus						
		1	2			1	2					
16(1–6)	√	√		11+12	√	√			√	√	√	√
(7–12)	√		√		√		√		√	√	√	√
17(1–6)		√			√	√				√		√
(7–12)			√		√		√			√		√
18(1–6)		√			√	√				√		√
(7–12)			√		√		√			√		√
19	√	√				√						√
20	√	√				√						√
21	√	√			√	√						√
22		√			√	√						√
23		√			√	√						√
24	√		√				√					√
25	√		√				√					√
26	√		√		√		√					√
27			√		√		√					√
28			√		√		√					√
Sq Rt						√					√	
Sw 1						√						√
Sw 2						√						√
Sw 3							√					√
Sw 4							√					√

The various relay groups performing the different functions can be plugged to operate on selected relay pulses as shown in figure 3. These operations include in and out for transfers; resetting; and starting to cross-add, multiply, divide, extract the square root, and interrupt a sequence (see §5).

The relays operate on a pentad system with a relay representing each digit from 1 to 5. In transferring a number from one group of relays to another, a separate conductor is used for each relay to permit simultaneous operation; five conductors are therefore required for one decimal digit, or 30 for a six-digit number. This mode of operation gives high speed (a single impulse of 0.0125 second is sufficient to set up a given relay configuration or to reset a group of relays) but increases the number of conductors and switches required.

		5				10				15				20				25				30				35				40				45			
15 to MC MP–MC Reset																																			
		X		X		X			X			X			X			X			X			X			X			X			X		
In–Out			X		X		X			X			X			X			X			X			X			X			X			X	
		X		X		X		X			X			X			X			X			X			X			X			X		
Reset																																			
Start	Ca 1–6 9–10(1) 11–12(1)	X					X				X				X				X				X				X				X				
	Ca 9–10(2) 11–12(2) Multiply					X		X				X				X				X				X				X				X			
	Divide Sq. Root Seq. Int.					X		X				X				X				X				X				X				X			

Figure 3

The actual operations of addition or subtraction require two impulses, but four impulses are required between successive additions or subtractions involving the same relay group. Addition or subtraction involves 3 relay groups, say A, B, and C, and the operation is as follows:

Impulse 1, 2: read quantities into A and B
3: test for carries
3, 4: read sum or difference into C
4: reset A and B
5, 6: read new quantities into A and B; transfer data in C elsewhere
6: clear C.

The arrangements of the relay groups for cross-adding are shown in table 1; the operation can take place simultaneously in several independent triads of groups. By using four groups (a, b, c, d), a series of numbers can be added at the rate of 40 per second; the operations are:

$$(a) + (b) \rightarrow c, \qquad c + (d) \rightarrow b, \qquad (a) + b \rightarrow c, \text{ etc.},$$

where new quantities read into the groups are indicated by parentheses.

Multiplication is performed by addition of left- and right-hand components in a manner analogous to that in the IBM Multiplying Punch (type 601); the details are different largely because of the relay method of addition. The right and left components are combined for each digit of the multiplier, and column shift is performed after the components are combined. Alternate sets of relay groups are employed for alternate digits of the multiplier. As shown in table 1, relay groups multiplier, multiplicand, numbers 1–8, and 16 are involved in the process which proceeds as follows (see table 2):

1. Read factors (MP and MC) into multiplier and multiplicand registers.
2. Form right- and left-hand components of multiplicand and first digit of multiplier (RC1 and LC1).
3. Add RC1 and LC1 to give the complete partial product of the multiplicand by the first digit of the multiplier, (p1). (Similarly form other partial products, p2 to p6.)
4. Combine the various partial products employing addition and column shift.

The operations of the registers are as shown in table 2 (note that each operation involves two impulses).

As shown in the table, successive multiplications may be spaced at intervals of $12 \times 0^s.0125 = 0^s.15$, provided full advantage is taken of overlap. A single multiplication without overlap requires .2 second.

Division is performed by subtracting the divisor, or five times the divisor, repeatedly from the dividend. The dividend is shifted to the left after the completion of the operation in each position. Subtraction is performed by adding the complement of the number to be subtracted. In adding the complement a carry will occur through the highest order, provided the number being subtracted is less than the one from which it is subtracted. This carry is sensed in advance so that the addition takes place only when desired; in other cases the shift of the dividend takes place instead. The

average dividing time is .2 second per quotient digit. There is capacity for 12 figures in the dividend, 6 in the divisor, and 10 in the quotient.

Square root is extracted by successive additions of odd numbers to the complement of the original number. There is capacity for 12 figures in the original number and 6 in the square root.

Table 2

Impulses	Registers										
	MP	MC	1	2	3	4	5	6	7	8	16
1 2	MP	MC	RC1	LC1							
3 4					RC2	LC2		p1			
5 6			RC3	LC3					p2	P1	
7 8					RC4	LC4	P2	p3			
9 10			RC5	LC5					p4	P3	
11 12					RC6	LC6	P4	p5			
13 14	MP	MC	RC1	LC1					p6	P5	
15 16					RC2	LC2	p1				P6

$$Pi = p1 + p2 + \cdots + pi$$

Numbers read from relay groups 13 and 16 may be shifted to the right or left. They are provided with pluggable output which permits any arbitrary shift; group 16 has two such sets of outlets. An automatic shift of 1, 2, or 3 columns to the right or left from group 16 may also be accomplished by plugging a single control wire.

Rounding of positive numbers may be accomplished in any adding group by adding 5. A credit balance may be rounded by wiring registers 11, 12, and 16 to add or subtract 5 according to the sign of the balance.

Card feed A and the relay comparing unit can be used for consulting mathematical tables. The table cards are fed at the rate of 100 cards per minute, so that long searches should be avoided by having the table cards in the same order or by using a short table with fairly long interpolation procedure. The first method is illustrated in §6. By combining field selection with card selection in a carefully constructed table, it is possible to consult a fairly extensive table in a few seconds.

5. Sequence Control

The basic operations of the machine are initiated by a pluggable sequence and to some extent by the punching on the cards. The latter include the

usual functions of identification of algebraic sign, the use of a digit emitter, rate card operation, and group control.

The pluggable sequence on the control panels initiates most of the operations. There are 96 sequencing relays with outlets to the plugboard so that impulses through them can be used for control purposes. These relays operate in succession, actuated by the successive relay impulses from the main timing circuit mentioned in a previous paragraph. The complete sequence of 96 relays is operated in two revolutions of the shaft, or a single cycle involving only 48 relays may be employed. When operating on the single-cycle basis, a card is fed during each revolution; on the double-cycle basis a card is fed on the first revolution of each successive pair of revolutions. The sequence can be temporarily suspended and resumed at the same point one revolution (48 impulses) later.

There are 25 selectors having a total of 132 circuits, with pluggable primary and secondary circuits. These selectors are used for combining and analyzing control impulses, and for routing digits under such control.

Operations such as multiplication, where the necessary number of cycles can be predicted in advance, can be plugged to give maximum efficiency of operation. In operations such as division, square root, table look-up, etc., where the required time cannot be predicted, the sequence is interrupted at the start of the operation and resumed when completed. By combining selectors with the sequencing relays it is possible to alter the sequence on successive revolutions so as to obtain a long sequence, limited only by the capacity of the relays available.

There are two control panels on the machine: one, of three sections, which in general determines the channels through which the data flow, and the other, of two sections, which determines the sequence of operations.

As shown in figure 2, numbers are read from, and punched on, the card during relay impulses 13 to 44; consequently, numbers read from a given card during impulses 13 to 44 may be used for computations during impulses 45 to 48 of that cycle and 1 to 12 of the next, and the result may be punched on the same card while the following card is being read. Since a multiplication requires 16 impulses, there is just time to perform a single multiplication on a card and record the answer on the same card (100 cards per minute). A number of cross-additions may be performed simultaneously, but there is no time to cross-add to the product. The sign of the product is determined in time for x or y punching.

Where the results need not be punched on the card from which the factors were read, it is possible to store them in the machine for the next card and thus use the entire card cycle for computation. In this manner it is possible to form 4 products and to do cross-adding at the rate of 100 cards per minute, with some limitations growing out of the available storage.

In comparatively long sequences, cards may be fed on every second cycle, third cycle, etc., the limit to the length of the sequence being determined by the internal storage capacity.

As an example of the control of the operations of the IBM Pluggable Sequence Relay Calculator, consider the problem of forming 4 products and their sum. The 8 factors are on one card in feed A, and the results are punched on another card in feed B. The operations require 2 cycles, or 96 impulses, and can be performed at the rate of 50 cards per minute.

Sixteen impulses after the multiplication is started, the product appears in register 16. It is possible to start a new multiplication every twelfth impulse if the multiplicand is transferred to register 15 and the "15 to MC" transfer is used.

The registers for the factors are selected to permit reading of all multipliers on bus 1 and all multiplicands on bus 2. The products are transferred from register 16 to a 12-digit register or to two 6-digit registers, where they are stored for punching during impulses (6–44) of the following cycle.

Table 3

Impulse	Instructions
6–44	Read at station "1" factors a_1, a_2, a_3, a_4 from card into registers MP, 19, 20, 21. Read at station "1" factors b_1, b_2, b_3, b_4 from card into registers MC, 24, 25, 26.
43	Start multiplication ($a_1 \times b_1 = P_1$).
46	Transfer b_2 to 15 (24 out, 15 in).
54	Reset MP, MC; transfer b_2 to MC (direct transfer of 15 to MC); transfer a_2 to MP (19 out, MP in).
55	Start multiplication ($a_2 \times b_2 = P_2$).
57	Reset 15.
58	Transfer b_3 to 15 (25 out, 15 in).
60	Transfer P_1 to 17 (16 out, 17 in).
64	Transfer P_1 to 12 (direct transfer of 16 to 12).
66	Reset MP, MC; transfer b_3 to MC (direct transfer of 15 to MC); transfer a_3 to MP (20 out, MP in).
67	Start multiplication ($a_3 \times b_3 = P_3$).
69	Reset 15, 16.
70	Transfer b_4 to 15 (26 out, 15 in).
72	Transfer P_2 to 18 and 11 (16 out, 18 in, 11 in).
73	Add P_1 and P_2 (cross-add 11 and 12); reset 16.
74	Transfer ($P_1 + P_2$) to 16 (direct transfer of (11 + 12) to 16).
75	Reset 11, 12.
76	Transfer ($P_1 + P_3$) to 12 (direct transfer of 16 to 12).
78	Reset MP, MC; transfer b_4 to MC (direct transfer of 15 to MC); transfer a_4 to MP (21 out, MP in).
79	Start multiplication ($a_4 \times b_4 = P_4$).
81	Reset 15, 16.
84	Transfer P_3 to (22 and 27) and 11 (16 out, 22 in, 27 in, 11 in).
85	Add P_3 to $P_1 + P_2$ (cross-add 11 and 12); reset 16.
86	Transfer ($P_1 + P_2 + P_3$) to 16 (direct transfer of (11 + 12) to 16).
87	Reset 11, 12.
88	Transfer ($P_1 + P_2 + P_3$) to 12 (direct transfer of 16 to 12).
90	Reset MP, MC.
93	Reset 16, 19, 20, 21, 24, 25, 26.
96	Transfer P_4 to (23 and 28) and 11 (16 out, 23 in, 28 in, 11 in).
1	Add P_4 to $P_1 + P_2 + P_3$ (cross-add 11 and 12); reset 16.
2	Transfer ($P_1 + P_2 + P_3 + P_4$) to 16 (direct transfer of (11 + 12) to 16).
3	Reset 11, 12.
6–44	Punch P_1, P_2, P_3, P_4, $\sum_1^4 P$; read new factors.
45	Reset 16, 17, 18, 22, 23, 27, 28.

Since a 12-digit sum is assumed, registers 11 and 12 are used to cross-add the products, and each sum is transferred into register 16 because there is a direct connection. After registers 11 and 12 are reset, the accumulated products are transferred back to register 12 for the next cross-addition. The final summation appears in register 16 just before punching time. Note that while the cards are being punched, the new entries are made for the following card (see figure 1). Table 3 indicates the necessary wiring.

6. Applications

The operation of the machines will be illustrated by a few typical examples: the reduction of observational data, the multiplication of matrices and of harmonic series, the solution of differential equations, and the construction of mathematical tables.

In many fields of science and engineering great masses of data are recorded and must be subjected to mathematical treatment in preparation for proper examination and discussion. The efficiency of the punched card for this work is well known and has been utilized in extensive programs at many computing laboratories.[1] Because of the ability of the relay calculators to eliminate collating, reproducing, and recording operations, they bring greatly increased efficiency to this field. In a recent program a series of 50,000 measurements, x and y, were corrected by the following formulae

$$\alpha = k + c + by + x(a - 0.016 + gy + px + 0.002x^2 + 0.002y^2)$$
$$\delta = k' + c' + a'x + y(b' - 0.016 + p'x + g'y + 0.002y^2 + 0.002x^2)$$

where

$$k, c, b, a, g, p$$
$$k', c', b', a', g', p'$$

are constant for several hundred values of x and y. The cards containing x and y were already completely punched; therefore a new set of cards was prepared during the computation. The values of the constants k, c, etc. were punched on master cards. The computation was performed in two runs by interchanging x and y on the control panel. The operation proceeds at the rate of 2,000 cards per hour.

A common type of reduction used in science is the following

$$E = Aa + Bb + Cc + Dd$$

where A, B, C, D are functions of one variable, x, and a, b, c, d are functions of another variable, y. In this case the cards containing the values of a, b, c, d, x would be sorted on x and placed in the punch feed; cards A, B, C, D would be placed in the read feed (sorted also on x). The machine would then treat the A, B, C, D cards as a mathematical table and perform the calculation as required.

The multiplication of matrices and of harmonic series are natural applications for these machines. The punched card gives unlimited storage capacity for large matrices or series, and the efficient reading, recording, and high-speed multiplication provide for high-speed operation.

ALT has described[2] a process whereby the cards containing the elements of the A matrix are placed in one feed, those of the B matrix in the other, and those of the product matrix C = AB are punched on the B cards. The process is continuous and proceeds at the rate of 6,000 cards per hour for 6×6 multiplication and 3,000 per hour for 12×12.

By inserting new cards for the C matrix it would be possible to perform 3 multiplications per card at the rate of 6,000 cards per hour as described in §4. This would permit 12-digit multiplication (small error in twelfth digit) or the multiplication of 3 separate pairs of matrices on the same cards.

These machines are the most efficient ones in operation for the multiplication of large harmonic series. Consider the case of such a series with

about 1,000 terms each, coefficients of 12 digits, and 3 or 4 fundamental frequencies.[3] In forming the product series there are a million possible terms, of which several thousand may have coefficients greater than the rejection limit and therefore will have to be retained. For each of these terms a product (of coefficients) and a sum (of arguments) must be formed. The relay calculator calculates such product terms automatically at the rate of 6,000 sensible terms per hour.

After the individual terms are formed, they are sorted according to the argument and combined to give the resultant series. This operation, which could be performed on the accounting machine and summary punch, is performed on the relay calculator. A linear combination of 2 or more series with some manipulation of the arguments requires a third machine set-up.

The relay calculators perform automatically the solution of ordinary linear differential equations up to and including the sixth order.[4] In the method used by HAUSMAN & SCHWARZSCHILD the elements of a sixth-order square matrix are computed, before the integration, for each value of the independent variable. The values of the 6 dependent variables for a given step constitute a matrix with one row of 6 elements. The integration procedure at each step consists of multiplying the square matrix by the single-row matrix to obtain a new single-row matrix whose elements are the values of the dependent variables for the next value of the independent variable.

In the first integration the fourth differences are neglected. After the first integration the resultant values are used to obtain corrections for these effects by a second run of the cards. On the second integration these corrections are included in the integration. The procedure is thus one of successive approximations of a rapidly converging process.

In applying the method to a set of the fifth order involving about 80 steps, it was found that each integration required about 12 minutes, and about one hour was required for the approximations to converge. The problem involved similar solutions for about 12 sets of initial conditions.

Although the machines have not been used for this purpose, they could be used for the solution of partial differential equations where each new value depends upon a number of values of the old ones. The standard punched-card machines have been used for such problems, and the relay calculators would provide greater speed and much less handling of the cards.

In the construction of mathematical tables the basic computations for a large number of values of the arguments are usually made at one time. These calculations are regularly performed on the standard machines; additional speed and freedom from card handling make the relay calculators efficient for such problems.

In a set of gear tables recently computed the following formulae were evaluated with one run of the cards, at the rate of about 1,000 values per hour.

$$x = AB/(C + D), \qquad z = xEF,$$
$$y = AB/(C + 10D), \qquad w = 10yEF,$$

where A, B, C, D, E, F are given numbers for each entry.

In another set of tables one step involved the solution of the transcendental equation

$$M = E - e \sin E$$

where M and e are given and E is required. A small punched-card table gave a rough approximation to E, E_0. This value E_0 was used to enter a standard punched-card table of sin E with tabular interval of 0°.01. From this table were taken sin E_0 and sin $(E_0 + 0°.01)$ by passing the cards through the reproducer in both directions. The relay calculator, with a single run of the cards, evaluated the following formulae:

$$\delta_1 = M - E_0 + 57°.2958e \sin E_0$$
$$\delta_2 = M - E_0 - 0°.01 + 57°.2958e \sin (E_0 + 0°.01)$$
$$E = E_0 - 0°.01\delta_1/(\delta_2 - \delta_1).$$

Several members of the Laboratory Staff have assisted in the preparation of this article, especially LILLIAN F. HAUSMAN, REBECCA JONES, MARJORIE HERRICK, PHYLLIS ARNOLD, and RICHARD BENNETT. Valuable suggestions were made by GEOFFREY KELLER of the Department of Astronomy of Columbia University.

W. J. E.

[1] W. J. ECKERT, *Punched Card Methods in Scientific Computation*, The Thomas J. Watson Astronomical Computing Bureau, 1940, chap. x.
[2] FRANZ L. ALT, "Multiplication of matrices," *MTAC*, v. 2, p. 12–13.
[3] W. J. ECKERT, *op. cit.*, chap. xi.
[4] L. F. HAUSMAN & M. SCHWARZSCHILD, "Automatic integration of linear sixth-order differential equations by means of punched card machines," *Rev. Sci. Inst.*, v. 18, 1947, p. 877–883.

Inversion of a Matrix of Order 38

The general solution of a system of 38 simultaneous linear equations has recently been obtained by the writer, utilizing the Aiken Relay Calculator, constructed for the Naval Proving Ground, Dahlgren, Virginia, by the staff of the Computation Laboratory of Harvard University. The elements of the reciprocal matrix were computed using an adaptation of the GAUSS method of elimination. This method is particularly suited to machine solution in that it permits the use of a few short computing routines, the number of repetitions of which are governed by simple functions of the system order. The fact that a single system of control tapes may be employed to invert a matrix of any order was considered of paramount importance in the choice of this method.

The remarkable feature of the present computation was the degree of accuracy obtained in the elements of the reciprocal matrix. When tested on known data, the inverse matrix yielded results correct to nine significant digits, notwithstanding the fact that the calculator employed is limited to ten significant digits.

The problem solved originated in the field of mathematical economics. It was posed by Professor WASSILY W. LEONTIEF [1] of the Department of Economics in Harvard University. The original system comprises the output-input relations among the industries of the United States, divided into 38 groups, as compiled by the United States Bureau of Labor Statistics for the year 1939. Thirty-eight equations arise, of the form,

$$(1) \qquad A_i - \sum_{j=1}^{38} B_{ij} = C_i, \qquad j \neq i,\ i = 1(1)38,$$

where

A_i = the output of the ith industry group,
B_{ij} = that portion of the output of group i which is used as input by group j, and
C_i = the "bill of goods," i.e., that portion of the ith group's output consumed, exported or unaccounted for.

For purposes of computation, each element in column j of the coefficients of the left-hand sides of equations (1) is normalized by dividing it by the diagonal element A_j, yielding

$$(2) \qquad \begin{aligned} a_{ij} &= B_{ij}/A_j, & i &\neq j, \\ a_{ij} &= A_i/A_j = 1, & i &= j. \end{aligned}$$

Assuming the normalized coefficients a_{ij} to be constant, prediction equations for the industry group outputs X_j,

$$(3) \qquad \sum_{j=1}^{38} a_{ij}X_j = Y_i, \qquad i = 1(1)38,$$

may be formed from the a_{ij} and any arbitrary but reasonable "bill of goods," Y_i. Each product $a_{ij}X_j$ represents that portion of the output of group i which is required as input by group j whatever the magnitude of X_j may be. By inverting the square matrix $[a]$ to form $[d] = [a]^{-1}$, the magnitudes, X_j, of the outputs required to produce the assumed "bill of goods," Y_i, may be found from the equations,

$$(4) \qquad X_j = \sum_{i=1}^{38} d_{ji}Y_i, \qquad j = 1(1)38.$$

Written in matrix notation, equations (3) and (4) become

$$(5) \qquad [a]\{X\} = [I]\{Y\},$$

$$(6) \qquad [I]\{X\} = [d]\{Y\},$$

where $[I]$ denotes the unit matrix of order 38.

The method of elimination accomplishes the transformation of equation (5) into equation (6) by introducing auxiliary matrices $[b]$ and $[c]$, which satisfy the relation,

$$(7) \qquad [b]\{X\} = [c]\{Y\},$$

where

$$[b] \equiv \begin{bmatrix} 1 & b_{1,2} & b_{1,3} & \cdots & b_{1,38} \\ 0 & 1 & b_{2,3} & \cdots & b_{2,38} \\ 0 & 0 & 1 & \cdots & b_{3,38} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}; \qquad [c] \equiv \begin{bmatrix} c_{1,39} & 0 & 0 & \cdots & 0 \\ c_{2,39} & c_{2,40} & 0 & \cdots & 0 \\ c_{3,39} & c_{3,40} & c_{3,41} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{38,39} & c_{38,40} & c_{38,41} & \cdots & c_{38,76} \end{bmatrix}.$$

Matrices $[a]$ and $[I]$ of equation (5) are transformed into $[b]$ and $[c]$, respectively, of equation (7) by the same operations as are used to reduce determinants, i.e., multiplication by a constant and subtraction from one row of a linear combination of other rows.[2] The same types of operations

are employed to convert $[b]$ and $[c]$ into $[I]$ and $[d]$, respectively, of equation (6).

Since the same operations are to be performed on the coefficient matrices of both sides of equations (5) and (7), it is convenient to combine the two matrices of each equation into a single augmented matrix. Thus $[a]$ and $[I]$ of equation (5) become a matrix $[a, I]$ of order 38×76, which will be referred to as the given augmented matrix. Similarly, $[b]$ and $[c]$ become auxiliary augmented matrix $[b, c]_{38,76}$, and $[I]$ and $[d]$ of equation (6) become the final augmented matrix $[I, d]_{38,76}$.

The transformation of the given augmented matrix into the auxiliary augmented matrix is accomplished by applying routines (I) and (II), as follows:

(I) the first row of the given augmented matrix is multiplied by the reciprocal of the leading element, yielding the first row of the auxiliary augmented matrix,

$$b_{1,j} = a_{1,1}^{-1} a_{1,j}, \qquad j = 1(1)76.$$

(II) each element of the remaining rows of the given augmented matrix is reduced by the product of the leading element of the row in question by the element of the corresponding column from the first row of the auxiliary augmented matrix, producing $[a^{(1)}]_{(n-1),2n}$

$$a_{i,j}^{(1)} = a_{i,j} - a_{i,1} b_{1,j}, \qquad i = 2(1)38, \qquad j = 1(1)76.$$

The result of routines (I) and (II) is that X_1 is eliminated from all but the first of equations (3).

The matrix $[a^{(1)}]$ is then subjected to routines (I) and (II), yielding the second row of the auxiliary augmented matrix $[a^{(2)}]_{(n-2),2n}$. The sequence of routines (I) and (II) is applied in turn to the residual matrices $[a^{(k)}]_{(n-k),2n}$ until all rows of the auxiliary augmented matrix have been computed.

The last row (38) of the auxiliary augmented matrix now defines explicitly the value of X_{38} (for given Y's), and hence the portion contained in row 38 of $[c]$ is identical with row 38 of $[d]$.

The remaining rows of the final augmented matrix are obtained from the auxiliary augmented matrix by applying routine (III), as follows:

(III) each element of the remaining rows of $[c]$ is reduced by the product of the last element of the corresponding row of $[b]$ by the element of the corresponding column from the last row of $[d]$, to produce $[c^{(1)}]_{(n-1),n}$,

$$c_{i,j}^{(1)} = c_{i,j} - b_{i,38} d_{38,j}, \qquad i = 37(-1)1, \qquad j = 39(1)76.$$

The last row (37) of $[c^{(1)}]$ is identical with row 37 of $[d]$.

Routine (III) is applied to each matrix $[c^{(k)}]_{(n-k),n}$ in turn, the product being formed by the $(n-k)$th element from the corresponding row of $[b]$ and the element of the corresponding column from the $(n-k)$th row of $[d]$. In each case, the last row of $[c^{(k)}]$ is identical with the corresponding row of $[d]$.

The application of this method of solution to large-scale calculators is greatly facilitated by the fact that the machine operations required to carry out routines (I) and (II) are the same for every row of the series of matrices

$[a]$, $[a^{(1)}]$, $\cdots$ $[a^{(m)}]$. The number of elements in each row is constant if those elements which are either 0 or 1 in the given and auxiliary augmented matrices are provided for by the coding rather than in the data tapes. Two very desirable features derive from this fact: (A) the control tapes may be short, iterative tapes, and (B) the solution can be made independent of the order n of the system. The application of routine (III) is complicated by the presence of the element $b_{i,\,(n-k)}$ in the routine, but features (A) and (B) can be retained.

The Aiken Relay Calculator is equipped with four typewriters (called printers), four tape punches, four tape readers, and four sequence control mechanisms. These are the limiting elements in the design of the computing routines, and in the present problem a reduction of one-third in the computing time would result from a 50 per cent increase in these components.

It was found necessary to use a pair of identical data tapes for storing the elements of all matrices, simultaneously printing the elements being punched to provide a reference in the event of disagreement between the paired tape values. The printing circuits of the calculator provide a check system to insure that the quantity printed is the same as the quantity registered in the machine. Similar circuits are not provided for in the tape punches since the data being punched can be checked by refeeding.

In carrying out the computation of the auxiliary augmented matrix, one pair of tape readers provided the input matrix by rows, and the associated tape punches produced the output matrix tapes. The row of elements computed by routine (I) was punched into a second pair of data tapes by the third and fourth tape punches; these tapes were then fed into the associated readers and were read and repunched continuously as the remaining rows of the matrix were calculated by routine (II). A similar disposition of the punches and readers was employed in the computation of the final inverse matrix.

Three sets of control tapes were employed in the solution. The first set of two performed the matrix multiplications implied by equations (2) and (4). The second set of four tapes produced the auxiliary augmented matrix. The computation of the final matrix required four more control tapes.

An appreciation of the magnitude of the computation can best be had from a few statistics. Although the application of routines (I), (II) and (III) required but three seconds per element, the total uninterrupted machine time needed to produce the solution was 59½ hours. The corresponding time for a matrix of order n would be approximately $0.001\ n^3$ hours. These estimates should be doubled to include the time required for set-up, tape changing, check re-runs and trouble shooting. Nearly 300000 ten-digit numbers were punched into and read from tapes, and 1000000 digits were printed. Approximately 100000 multiplications and 600000 additions were performed.

The present problem was the first serious computation to be attempted on the calculator, and its solution was beset by all the difficulties inevitably associated with the prototype of a new design. Overcoming these difficulties resulted in (a) improvements in the calculator amounting to at least 25 per cent greater operating efficiency; (b) uncovering a number of mistakes in wiring and other errors of construction prior to shipment to the Naval Proving Ground; (c) development of operating and trouble-shooting procedures; and (d) familiarization of all personnel—mathematicians, operators, and tech-

TABLE: *Comparison of Original Data with Those Computed from the Inverse Matrix*

i	Original Data A_i	Computed Values X_i	Absolute Error $\times 10^{-9}$	Relative Error $\times 10^{-9}$
1	10121	10120.999 99	1	1
2	13282	13281.999 91	9	7
3	2622	2622.000 001	0.1	0.4
4	493	492.999 999 7	.03	.6
5	438	437.999 999 8	.02	.4
6	443	443.000 000 3	.03	.6
7	134	133.999 999 1	.09	7
8	2581	2580.999 999	.1	0.4
9	269	269.000 000 1	.01	.4
10	266	266.000 000 5	.05	2
11	2223	2222.999 993	.7	3
12	441	440.999 999 3	.07	2
13	331	331.000 000 3	.03	1
14	1853	1852.999 997	.3	2
15	2287	2287.000 005	.5	2
16	1616	1615.999 999	.1	0.6
17	2082	2081.999 993	.7	3
18	4871	4871.000 017	1.7	4
19	1737	1737.000 001	0.1	0.6
20	2864	2864.000 007	.7	3
21	1516	1515.999 999	.1	0.7
22	3434	3434.000 003	.3	1
23	1245	1245.000 001	.1	0.8
24	1188	1187.999 999	.1	.8
25	1712	1711.999 997	.3	2
26	2265	2264.999 999	.1	0.4
27	3187	3187.000 003	.3	1
28	3454	3454.000 005	.5	2
29	990	989.999 998 9	.11	1
30	892	891.999 999 3	.07	0.8
31	1666	1665.999 999	.1	0.6
32	10089	10089.000 01	1	1
33	3167	3167.000 007	0.7	2
34	4310	4310.000 001	.1	0.2
35	16571	16571.000 01	1	.6
36	18525	18524.999 99	1	.5
37	4152	4152.000 005	0.5	1
38	22192	22191.999 97	3	1

$$X_i = \sum_{k=1}^{38} d_{ik} C_k, \qquad i = 1(1)38,$$

where X_i = computed total output of industry group i;
d_{ik} = computed coefficients of the inverse matrix $[d]$;
C_k = "bill of goods" actually consumed or exported during the year 1939 (not previously utilized in the computation);
and A_i = actual output of industry group i for the year 1939.

nicians—with the capabilities, limitations, and operating characteristics of the calculator and its associated equipment.

Three levels of checks were employed in the computation. The machine operations required for each element were performed in duplicate, using distinct machine components, and the results compared before they were printed or punched. A check number for each row of the given augmented matrix was formed during the normalization routine by summing the elements in the row, the check number sum becoming the 77th element. The completed rows of the auxiliary and final matrices were then summed, and that sum compared with the modified check number. A complete system of checks was found necessary for all control numbers used to regulate the repetitions of the control tapes. Finally, the original right-hand-side elements C of equations (1) were utilized as the Y's of equations (4). The X's obtained from the solution of equations (4) were then compared with the original A's of equations (1). The comparisons are given in the table.

The remarkably small differences resulting from these comparisons are attributed to two factors. First, the adders and multipliers of the calculator are equipped with round-off compensation circuits; and, second, each row of the given matrix possessed several dominant coefficients, the remaining ones being quite small in comparison. Round-off errors in the latter were largely absorbed, in the summing process, by the larger terms from the dominant coefficients.

Recently von NEUMANN & GOLDSTINE[3] have shown that the loss of significant digits to be expected, under certain rather general assumptions, when numerically inverting an arbitrary large matrix, is very great. However, as the results of the present computation show, the presence of a relatively small number of dominant coefficients in the given matrix may materially reduce this loss. Under this special condition, it may be expected that the inverse of a high-order matrix derived from a stable physical system will be as accurate as the input data. The rapid development of large-scale digital calculating machines should soon provide more examples of this operation, supplying further information concerning the relation between the number of dominant coefficients and the loss of significant digits.

The writer wishes to express his appreciation to Professor LEONTIEF and to Mr. FREDERICK MILLER and the operating staff of the calculator, without whose conscientious efforts the solution of the problem could not have been obtained.

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Coefficients for Expressing the First Thirty Powers in Terms of the Hermite Polynomials[1]

The present table of coefficients gives the exact expression for x^n, $n = 0(1)30$, in terms of the HERMITE polynomials, $H_m(x)$, where $H_m(x) \equiv (-1)^m e^{x^2}(d^m e^{-x^2}/dx^m)$. The computation followed the formula given in E. FELDHEIM, "Formules d'inversion et autres relations pour les polynomes orthogonaux classiques," Soc. Math. d. France, *Bull.*, v. 68, p. 199–228, namely,

$$x^n = 2^{-n} \sum_{\nu=0}^{[\frac{1}{2}n]} n!/\{\nu!(n - 2\nu)!\} H_{n-2\nu}(x).$$

The quantities $n!/\{\nu!(n - 2\nu)!\}$ were expressed as $\binom{n}{2\nu}(2\nu)!/\nu!$ After obtaining the factors $(2\nu)!/\nu!$, they were multiplied by $\binom{n}{2\nu}$, which were taken from J. W. L. GLAISHER's table of binomial coefficients (*Mess. Math.*, v. 47, p. 97–107). An overall check on the final manuscript was provided by evaluating the right member of each of the expressions for x^n that are given below, for $x = 1$. The necessary auxiliary quantities $H_m(1)$ were obtained from the recurrence formula $H_{n+1}(1) = 2H_n(1) - 2n\,H_{n-1}(1)$.

Similar to the use for the expression of powers in terms of LEGENDRE polynomials, this present table is useful for approximating a polynomial of high degree by a polynomial of much lower degree, which is best in an important least square sense (interval $-\infty$ to $+\infty$, weight factor e^{-x^2}). Thus when any function $f(x)$, in the interval $[-\infty, \infty]$, is expanded in terms of Hermite polynomials (where the coefficients in that expansion are determined by the condition of orthogonality with weight factor e^{-x^2} in the interval $[-\infty, \infty]$), if the partial sum of degree r is denoted by $q_r(x)$, while any polynomial of degree $\leqslant r$ which is distinct from $q_r(x)$ is denoted by $p_r(x)$, then

$$\int_{-\infty}^{\infty} e^{-x^2}[f(x) - q_r(x)]^2 dx < \int_{-\infty}^{\infty} e^{-x^2}[f(x) - p_r(x)]^2 dx.$$

The analogous set of coefficients for the expansion of x^n in terms of the LAGUERRE polynomials $L_m(x) \equiv (e^x/m!)d^m(e^{-x}x^m)/dx^m$ is useful for best approximation in the least square sense, where the interval is $[0, \infty]$ and the weight factor is e^{-x}. However it is unnecessary to compute them, since from the expression

$$x^n = n! \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{n-\nu} L_\nu(x),$$

it is obvious that all one needs are the binomial coefficients, and Glaisher's readily available table provides the coefficients up to x^{30}. Coefficients up to

[1] The coefficient of H_m in the expression for $(2x)^n$ is identical (except for alternation of + and − sign) with the coefficient of $(2x)^m$ in $H_n(x)$. Thus there are given here also the first thirty Hermite polynomials.

x^{60} may be had in the appendix, by J. PETERS & J. STEIN, to J. Peters, *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922.

$$x^0 = H_0$$

$$2\,x^1 = H_1$$

$$4\,x^2 = H_2 + 2H_0$$

$$8\,x^3 = H_3 + 6H_1$$

$$16\,x^4 = H_4 + 12\,H_2 + 12\,H_0$$

$$32\,x^5 = H_5 + 20\,H_3 + 60\,H_1$$

$$64\,x^6 = H_6 + 30\,H_4 + 180\,H_2 + 120\,H_0$$

$$128\,x^7 = H_7 + 42\,H_5 + 420\,H_3 + 840\,H_1$$

$$256\,x^8 = H_8 + 56\,H_6 + 840\,H_4 + 3360\,H_2 + 1680\,H_0$$

$$512\,x^9 = H_9 + 72\,H_7 + 1512\,H_5 + 10080\,H_3 + 15120\,H_1$$

$$1024\,x^{10} = H_{10} + 90\,H_8 + 2520\,H_6 + 25200\,H_4 + 75600\,H_2 + 30240\,H_0$$

$$2048\,x^{11} = H_{11} + 110\,H_9 + 3960\,H_7 + 55440\,H_5 + 2\ 77200\,H_3 + 3\ 32640\,H_1$$

$$4096\,x^{12} = H_{12} + 132\,H_{10} + 5940\,H_8 + 1\ 10880\,H_6 + 8\ 31600\,H_4 + 19\ 95840\,H_2 + 6\ 65280\,H_0$$

$$8192\,x^{13} = H_{13} + 156\,H_{11} + 8580\,H_9 + 2\ 05920\,H_7 + 21\ 62160\,H_5 + 86\ 48640\,H_3 + 86\ 48640\,H_1$$

$$16384\,x^{14} = H_{14} + 182\,H_{12} + 12012\,H_{10} + 3\ 60360\,H_8 + 50\ 45040\,H_6 + 302\ 70240\,H_4 + 605\ 40480\,H_2 + 172\ 97280\,H_0$$

$$32768\,x^{15} = H_{15} + 210\,H_{13} + 16380\,H_{11} + 6\ 00600\,H_9 + 108\ 10800\,H_7 + 908\ 10720\,H_5 + 3027\ 02400\,H_3 + 2594\ 59200\,H_1$$

$$65536\,x^{16} = H_{16} + 240\,H_{14} + 21840\,H_{12} + 9\ 60960\,H_{10} + 216\ 21600\,H_8 + 2421\ 61920\,H_6 + 12108\ 09600\,H_4 + 20756\ 73600\,H_2 + 5189\ 18400\,H_0$$

$$1\ 31072\,x^{17} = H_{17} + 272\,H_{15} + 28560\,H_{13} + 14\ 85120\,H_{11} + 408\ 40800\,H_9 + 5881\ 07520\,H_7 + 41167\ 52640\,H_5 + 1\ 17621\ 50400\,H_3 + 88216\ 12800\,H_1$$

$$2\ 62144\,x^{18} = H_{18} + 306\,H_{16} + 36720\,H_{14} + 22\ 27680\,H_{12} + 735\ 13440\,H_{10} + 13232\ 41920\,H_8 + 1\ 23502\ 57920\,H_6 + 5\ 29296\ 76800\,H_4 + 7\ 93945\ 15200\,H_2 + 1\ 76432\ 25600\,H_0$$

$$5\ 24288\,x^{19} = H_{19} + 342\,H_{17} + 46512\,H_{15} + 32\ 55840\,H_{13} + 1269\ 77760\,H_{11} + 27935\ 10720\,H_9 + 3\ 35221\ 28640\,H_7 + 20\ 11327\ 71840\,H_5 + 50\ 28319\ 29600\,H_3 + 33\ 52212\ 86400\,H_1$$

$$10\ 48576\,x^{20} = H_{20} + 380\,H_{18} + 58140\,H_{16} + 46\ 51200\,H_{14} + 2116\ 29600\,H_{12} + 55870\ 21440\,H_{10} + 8\ 38053\ 21600\,H_8 + 67\ 04425\ 72800\,H_6 + 251\ 41596\ 48000\,H_4 + 335\ 22128\ 64000\,H_2 + 67\ 04425\ 72800\,H_0$$

$$20\ 97152\,x^{21} = H_{21} + 420\,H_{19} + 71820\,H_{17} + 65\ 11680\,H_{15} + 3418\ 63200\,H_{13} + 1\ 06661\ 31840\,H_{11} + 19\ 55457\ 50400\,H_9 + 201\ 13277\ 18400\,H_7 + 1055\ 94705\ 21600\,H_5 + 2346\ 54900\,H_3 + 1407\ 92940\ 28800\,H_1$$

$$41\ 94304\,x^{22} = H_{22} + 462\,H_{20} + 87780\,H_{18} + 89\ 53560\,H_{16} + 5372\ 13600\,H_{14} + 1\ 95545\ 75040\,H_{12} + 43\ 02006\ 50880\,H_{10} + 553\ 11512\ 25600\,H_8 + 3871\ 80585\ 79200\,H_6 + 12906\ 01952\ 64000\,H_4 + 15487\ 22343\ 16800\,H_2 + 2815\ 85880\ 57600\,H_0$$

$$83\ 88608\,x^{23} = H_{23} + 506\,H_{21} + 1\ 06260\,H_{19} + 121\ 13640\,H_{17} + 8237\ 27520\,H_{15} + 3\ 45965\ 55840\,H_{13} + 89\ 95104\ 51840\,H_{11} + 1413\ 51642\ 43200\,H_9 + 12721\ 64781\ 88800\,H_7 + 59367\ 68982\ 14400\,H_5 + 1\ 18735\ 37964\ 28800\,H_3 + 64764\ 75253\ 24800\,H_1$$

$$167\ 77216\,x^{24} = H_{24} + 552\,H_{22} + 1\ 27512\,H_{20} + 161\ 51520\,H_{18} + 12355\ 91280\,H_{16} + 5\ 93083\ 81440\,H_{14} + 179\ 90209\ 03680\,H_{12} + 3392\ 43941\ 83680\,H_{10} + 38164\ 94345\ 66400\,H_8 + 2\ 37470\ 75928\ 57600\,H_6 + 7\ 12412\ 27785\ 72800\,H_4 + 7\ 77177\ 03038\ 97600\,H_2 + 1\ 29529\ 50506\ 49600\,H_0$$

$$335\ 54432\ x^{25} = H_{24} + 600\ H_{23} + 1\ 51800\ H_{22} + 212\ 52000\ H_{21} + 18170\ 46000\ H_{17} + 9\ 88473\ 02400\ H_{15} + 345\ 96555\ 84000\ H_{13} + 7710\ 08958\ 72000\ H_{11} + 1\ 06013\ 73182\ 40000\ H_9 + 8\ 48109\ 85459\ 20000\ H_7 + 35\ 62061\ 38928\ 64000\ H_5 + 64\ 76475\ 25324\ 80000\ H_3 + 32\ 38237\ 62662\ 40000\ H_1$$

$$671\ 08864\ x^{26} = H_{25} + 650\ H_{24} + 1\ 79400\ H_{23} + 276\ 27600\ H_{22} + 26246\ 22000\ H_{15} + 16\ 06268\ 66400\ H_{16} + 642\ 50746\ 56000\ H_{14} + 16705\ 19410\ 56000\ H_{12} + 2\ 75635\ 70274\ 24000\ H_{10} + 27\ 56357\ 02742\ 40000\ H_8 + 154\ 35599\ 35357\ 44000\ H_6 + 420\ 97089\ 14611\ 20000\ H_4 + 420\ 97089\ 14611\ 20000\ H_2 + 64\ 76475\ 25324\ 80000\ H_0$$

$$1342\ 17728\ x^{27} = H_{27} + 702\ H_{25} + 2\ 10600\ H_{23} + 355\ 21200\ H_{21} + 37297\ 26000\ H_{19} + 25\ 51132\ 58400\ H_{17} + 1156\ 51343\ 80800\ H_{15} + 34695\ 40314\ 24000\ H_{13} + 6\ 76560\ 36127\ 68000\ H_{11} + 82\ 69071\ 08227\ 20000\ H_9 + 595\ 37311\ 79235\ 84000\ H_7 + 2273\ 24281\ 38900\ 48000\ H_5 + 3788\ 73802\ 31500\ 80000\ H_3 + 1748\ 64831\ 83769\ 60000\ H_1$$

$$2684\ 35456\ x^{28} = H_{28} + 756\ H_{26} + 2\ 45700\ H_{24} + 452\ 08800\ H_{22} + 52216\ 16400\ H_{20} + 39\ 68428\ 46400\ H_{18} + 2023\ 89851\ 66400\ H_{16} + 69390\ 80628\ 48000\ H_{14} + 15\ 78640\ 84297\ 92000\ H_{12} + 231\ 53399\ 03036\ 16000\ H_{10} + 2083\ 80591\ 27325\ 44000\ H_8 + 10608\ 46646\ 48202\ 24000\ H_6 + 26521\ 16616\ 20505\ 60000\ H_4 + 24481\ 07645\ 72774\ 40000\ H_2 + 3497\ 29663\ 67539\ 20000\ H_0$$

$$5368\ 70912\ x^{29} = H_{29} + 812\ H_{27} + 2\ 85012\ H_{25} + 570\ 02400\ H_{23} + 72108\ 03600\ H_{21} + 60\ 57075\ 02400\ H_{19} + 3452\ 53276\ 36800\ H_{17} + 1\ 34155\ 55881\ 72800\ H_{15} + 35\ 21583\ 41895\ 36000\ H_{13} + 610\ 40779\ 26186\ 24000\ H_{11} + 6714\ 48571\ 88048\ 64000\ H_9 + 43949\ 36106\ 85409\ 28000\ H_7 + 1\ 53822\ 76373\ 98932\ 48000\ H_5 + 2\ 36650\ 40575\ 36819\ 20000\ H_3 + 1\ 01421\ 60246\ 58636\ 80000\ H_1$$

$$10737\ 41824\ x^{30} = H_{30} + 870\ H_{28} + 3\ 28860\ H_{26} + 712\ 53000\ H_{24} + 98329\ 14000\ H_{22} + 90\ 85612\ 53600\ H_{20} + 5754\ 22127\ 28000\ H_{18} + 2\ 51541\ 67278\ 24000\ H_{16} + 75\ 46250\ 18347\ 20000\ H_{14} + 1526\ 01948\ 15465\ 60000\ H_{12} + 20143\ 45715\ 64145\ 92000\ H_{10} + 1\ 64810\ 10400\ 70284\ 80000\ H_8 + 7\ 69113\ 81869\ 94662\ 40000\ H_6 + 17\ 74878\ 04315\ 26144\ 00000\ H_4 + 15\ 21324\ 03698\ 79552\ 00000\ H_2 + 2\ 02843\ 20493\ 17273\ 60000\ H_0.$$

HERBERT E. SALZER

NBSCL

RECENT MATHEMATICAL TABLES

519[A].—A. J. SACHS, "Babylonian mathematical texts, I. Reciprocals of regular sexagesimal numbers," *Jn. Cuneiform Studies*, v. 1, 1947, p. 219–240. 21.5 × 27.9 cm.

Regular sexagesimal numbers are those whose reciprocals may be expressed in a finite number of terms. In Old-Babylonian table texts (say, 1700 B.C.) the object of a set of multiplication tables was not only to yield the results of any multiplication, but also to give the multiples of reciprocals commonly used in division. "The existence of a multiplication table for the three-place number 44, 26, 40, for example, makes sense only in the light of the fact that 44, 26, 40 is the reciprocal of 1, 21."

Dr. Sachs has established the standard technique employed in the Old-Babylonian period to find the reciprocal of any regular number which is not contained in the standard reciprocal table (say, 2, 5 or 23, 43, 49, 41, 15).

Suppose c denotes a regular number whose reciprocal one wishes to find. Then choose two numbers a and b such that $c = a + b$ and a is a number which is found in the standard table of reciprocals. Using the ordinary notation $\bar{n} = 1/n$, it is evident that $\bar{c} = \bar{a}(\overline{1 + b\bar{a}})$.

In illustration of this analysis Dr. Sachs gives a transcription (p. 237) of the 21 paragraphs of tablet CBS 1215 of the Philadelphia collection. This is a table of the reciprocals of $2^n(2, 5)$ for $n = 0(1)20$, [that is 125 to 131 072 000], with the intermediate steps of the work; also for the reciprocals of the final results, showing that the number started with is found. All this material is rearranged as a table (p. 238–240), 7 columns on each page, headed c, b, a, $\bar{a}$, $1 + \bar{a}b$, $\overline{1 + \bar{a}b}$, $\bar{c}$. For example, in no. 1, $c = 2; 5$, $\bar{c}$ is found to be 0; 28, 48. In no. 19, second part, $c = 0; 0, 0, 0, 23, 43, 49, 41, 15$, and $\bar{c}$ is found to be 2, 31, 42, 13; 20. In each part of the latter example three applications of the formula are necessary, after reaching the column $1 + \bar{a}b$.

This paper is the first of a series of articles dealing with unpublished Babylonian mathematical texts, chiefly at the University Museum in Philadelphia.

R. C. A.

520[A, J].—JOHN Q. STEWART, "Empirical mathematical rules concerning the distribution and equilibrium of population." *Geographical Review*, v. 37, 1947, p. 461–485. 17 × 25.5 cm.

On p. 465 is a 2D table, 15 of the 29 different values of which were given in more extended form by J. W. L. GLAISHER, "On the constants that occur in certain summations by Bernoulli's series," London Math. Soc., *Proc.*, v. 4, 1872, p. 55. These 15 values are of $\Phi(-1/m)$ for $m = 1(1)5, 10$, and $\Phi(-1 - 1/m)$, for $m = 1(1)9$, each to 6D. The remaining 14 values may be calculated from $\Phi(-1 - 1/m)$ for $m = 3/4$ and 2; and from $\Phi(-1/m)$, for $1/m = .3, .4, .6, .7, .9, \frac{3}{2}, \frac{3}{4}, \frac{4}{3}, \frac{5}{2}, \frac{5}{4}, \frac{6}{5}, \frac{7}{5}, \frac{10}{9}$. $\Phi(-1 - 1/m) = 1^{-1-1/m} + 2^{-1-1/m} + 3^{-1-1/m} + \cdots$ and $\Phi(-1/m) = 1^{-1/m} + 2^{-1/m} + \cdots + x^{-1/m} - mx^{1-1/m}/(m - 1) - \frac{1}{2}x^{-1/m} + x^{-1-1/m}/12m - (m + 1)(2m + 1)x^{-3-1/m}/720m^3 + \cdots, x = 10$.

R. C. A.

521[B, C, D, E, F, H].—J. H. LAMBERT, *Opera Mathematica. Volumen secundum: Commentationes Arithmeticae Algebraicae et Analyticae, pars altera.* Zürich, Füssli, 1948, xxx, ii, 324 p. 15.5 × 22.9 cm.

This second volume of the works of Lambert contains the conclusion of his work in the fields of Arithmetic, Algebra and Analysis. In reviewing the first volume (*MTAC*, v. 2, p. 339–341) we inadvertently quoted the German title page on the dust wrapper instead of the Latin title page of the volume. In the volume under review there are only three parts of interest to us because of tabular material.

I. We shall first consider Lambert's memoir on hyperbolic functions, "Observations trigonométriques," p. 245–269, presented to the Berlin Academy of Sciences in 1768; the table is on page 268–269. The notation we are going to use is explained elsewhere in this issue, in N94, on Lambertian Functions. The hyperbolic sector $u = \ln \tan (45° + \frac{1}{2}\omega)$, $x^2 - y^2 = 1$ or $x = \cosh u$, $y = \sinh u$ being equations of the hyperbola. For the circle the corresponding equations are $x^2 + y^2 = 1$ or $x = \cos \phi$, $y = \sin \phi$; $\tan \phi = \sin \omega$, $\tan \omega = \sinh u$, $\sec \omega = \cosh u$. In the table are 9 columns; the first is $\omega = 0(1°)90°$; the second column contains the values of the corresponding hyperbolic sector u (according to Lambert). Lambert really gives the values of $\log \tan (45° + \frac{1}{2}\omega)$, so that his values must be multiplied by 2.30258509 in order to give the true approximate values of the hyperbolic sectors u corresponding to successive values of ω. It is curious that Professor SPEISER overlooked this Lambert error which makes the values in the following four succeeding columns of the table impossible to check: $\sinh u$ and $\log \sinh u$ (cols. 3, 5), $\cosh u$ and $\log \cosh u$ (cols. 4, 6). Column 9 gives the values of ϕ to the nearest tenth of a second corresponding to each value

of ω; columns 7 and 8 give respectively the values of tan ϕ and log tan ϕ. The values in cols. 2–8 are all to 7D. The erroneous col. 2 appears also in Lambert's interesting volume of tables,

II. *Zusätze zu den logarithmischen und trigonometrischen Tabellen zur Erleichterung und Abkürzung der bey Anwendung der Mathematik vorfallenden Berechnungen.* Berlin, 1770, 4, 98, 210 p. We have noted some facts concerning the origin of this volume in our previous review (*l.c.*, p. 340). In the volume before us it occupies p. 1–69, 75–111 + a title-page leaf. It contains 44 tables which are reproduced in full except for 8, namely: T.I, the smallest divisors of numbers not divisible by 2, 3 or 5, from 1 to 102000, filling p. 2–69 of the original, only p. 2–3 are reprinted; T.VI, prime numbers 1–102000 (p. 73–117), only p. 73; T.XV, hyperbolic logarithms of 1.01(.01)10 to 7D (p. 125–133), only p. 125; T.XXV, $n \sin A$, $n = 1(1)9$, $A = [1°(1°)90°; 5D]$ (p. 152–157), only p. 152–153; T.XXXII, hyperbolic functions (p. 176–181), only p. 176–177; T.XXXV, N^3, $N = 1(1)1000$ (p. 184–189), only 184–185; T.XXXVI, N^3, $N = 1(1)1000$ (p. 190–195), only p. 190–191; T.XL, N^p, $p = 1(1)11$, $N = .01(.01)1$ (p. 202–207), only p. 202–203. All other tabular pages are printed in facsimile. Since the volume is so rare it is too bad that it was not completely reproduced. Some of the tables not previously mentioned are the following: T.VII–IX, of 2^n, $n = 1(1)70$, 3^n and 5^n, each for $n = 1(1)50$. T.XIX gives the exact values of $\sin 3n°$, $n = 1(1)30$. T.XXIII, lengths of arcs of a unit circle to 27D, subtended by central angles $1°(1°)100°$, $120°(30°)270°$, $330°$, $360°$. T.XXIX, for solving the equations $\pm x \mp x^2 = a$ (see *MTAC*, v. 2, p. 28–29). T.XXXVII, 30 terms of the first 12 figurate numbers: x, $\frac{1}{2}x(x+1)$, $\frac{1}{3}x(x+1)(x+2)$, etc. T.XLII, various rational approximations to $N^{\frac{1}{2}}$, $N = 2(1)12$; T.XLIV, decimal representation of the coefficients of the first 15 terms in $(1+x)^{+\frac{1}{2}}$, and $(1+x)^{-\frac{1}{2}}$.

A posthumous Latin edition of Lambert's Tables was published in 1798 under the direction of Anton Felkel, by the Royal Academy of Sciences at Lisbon: *Supplementa Tabularum Logarithmicarum et Trigonometricarum auspiciis Almae Academiae Regiae Scientiarum Olisiponensis cum versione introductionis germanicae in latinum sermonem secundum ultima auctoris consilia amplificata.* lxxvi, 203 p. + plate. In T.I, where Lambert gave only the least factor, Felkel provides all prime factors except the greatest, the least figures being printed in figures, and the others denoted by letters. The erroneous column in T.XXXII is perpetuated here.

III. We come now to p. 70–73, taken from the preface of Lambert's *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin, 1772. There is here a table of 70 corrections, found by Wolfram, for T.I of Lambert's *Zusätze*. These have all been corrected in the *Supplementa*. It is also noted that in T.VI the prime number 91183 is missing; this is also wanting in the *Supplementa*. Various other corrigenda for other parts of the *Zusätze* are also listed.

Lambert's appeals to others, for calculating large factor tables, resulted in activity on the part of five persons, Oberreit, von Stamford, Rosenthal, Felkel, and Hindenburg. The correspondence in this connection is printed in the fifth volume of *Joh. Heinrich Lamberts . . . deutsche gelehrter Briefwechsel. Herausgegeben von Joh. Bernoulli*, Berlin, 1785. Felkel alone has left any record of his work; see *Scripta Mathematica*, v. 4, 1936, p. 336–337.

Lambert refers briefly to a number of tables, concerning which we may give further information in a few notes.

Büchner's table of squares and cubes from 1 to 12000 (Lambert, v.2, p. 4, 7, 8, 58, 70). J. P. Büchner, *Tabula Radicum*, Nuremberg, 1701.

Joncourt published a table of *Trigonalzahlen* (Lambert, v. 2, p. 4). This is É. de Joncourt, *Traité sur la Nature et sur les Principaux Usages de la Plus Simple Espèce de Nombres Trigonaux*. The Hague, 1762. Also Latin edition *De Natura et Praeclaro Usu simplicissimae speciei Numerorum Trigonalium*, The Hague, 1862, viii, 28, 224, 7 p. (This is the book which Prof. Speiser identifies as "Joncourt: *de la nature des nombres trigonaux*, 1742.")

Krüger-Jäger list of primes (Lambert, v. 1, p. 117; v. 2, p. 6, 7, 8, 14). J. G. Krüger, *Gedanken von der Algebra nebst den Primzahlen von 1 bis 1000000 . . .*, Halle, 1746. Algebra, 124 p.; list of primes, 47 p. The primes are from 1 to 100999, the title being incorrect. Lambert tells us (v. 2, p. 6) that Krüger received the table from Peter Jäger.

Anjema factor table from 1 to 10000 (Lambert, v. 1, p. 117–118; v. 2, p. 5, 7, 23). See *MTAC*, v. 2, p. 340, 374.

Poetius (Lambert, v. 1, p. 117, 118; v. 2, p. 5, 7, 14). J. M. POETIUS, *Rekenkunst mit den parallelen Algebra, Arithmetica*. Appendix, "anatomica numerorum." Leipzig, 1728. This appendix of primes and divisors up to 10000 is in [G. F. RICHTER] *Vollständigen mathematischen Lexici, zweiter Theil*. Leipzig, 1742, p. 530–543.

"Pell's" factor table (Lambert, v. 1, p. 118; v. 2, p. 5, 6, 7, 11, 14). Lambert erroneously attributes to Pell a factor table up to 100 000 (multiples of 2 and 5 omitted), in the compilation of which Pell had no part. It is interesting that in v. 2, p. 70, Lambert refers to his own factor table as a "Pell table." The facts are as follows: A Swiss, J. H. RAHN, wrote a work, *Teutsche Algebra oder Algebraische Rechenkunst*, Zürich, 1659, of which there was an English translation: *An Introduction to Algebra. Translated out of the High-Dutch into English, by Thomas Brancker . . . Much Altered and Augmented by D. P.* [Dr. John Pell]. *Also A Table of Odd Numbers less than One Hundred Thousand, showing Those that are Incomposit, and Resolving the rest into their Factors . . . Supputated by the same Tho. Brancker.* London, 1668, viii, 198, 50 p. + 1 plate. Thus we see by the title page itself that the original table of Rahn up to 24000 was extended by THOMAS BRANCKER up to 100 000. Also, the table is headed: "Tho. Branker's (*sic*) Table of Incomposit numbers, less than 100 000." Brancker's table is reprinted in F. MASERES, *The Doctrine of Permutations and Combinations* . . ., London, 1795, p. 366–416. From 20000 to 100000 it is also reprinted in E. HINKLEY, *Tables of the Prime Numbers, and Prime Factors of the Composite Numbers, from 1 to 100 000*. Baltimore, 1853, p. 165–205.

The reader interested in fuller information on early factor tables should turn to JAMES GLAISHER, *Factor Table for the Fourth Million*, London, 1879, p. 17–28; and J. W. L. GLAISHER, "On factor tables, with an account of the mode of formation of the factor table for the fourth million," Camb. Phil. Soc., *Proc.*, v. 3, 1878, p. 99–138.

In our previous review we referred (*MTAC*, v. 2, p. 341) to results of WALLIS (1685) giving the values of convergents of 17 terms of the continued fraction for $1/\pi = \frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\frac{1}{1}+\frac{1}{292}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{21}+\frac{1}{2}+\frac{1}{13}+\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{84}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots$, that is, through the second $\frac{1}{2}$ before $\frac{1}{84}$. I have since noticed that WOLFRAM in a letter to Lambert on 3 Aug. 1772 gave (87 years after Wallis) this same value (*Joh. Heinrich Lambert deutscher gelehrter Briefwechsel* hrsg. JOH. BERNOULLI, v. 4, 1784, p. 445–450) and also the next convergent, for the first $\frac{1}{2}$ before $\frac{1}{84}$, namely:

18 50401 87797 33719 17511:5 89001 21117 19765 29866

$= 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 6897$, which is correct to 41D.

R. C. A.

522[D].—J. PETERS, *Sechsstellige Tafel der trigonometrischen Funktionen enthaltend die Werte der sechs trigonometrischen Funktionen von zehn zu zehn Bogensekunden des in 90° geteilten Quadranten und die Werte der Kotangente und Kosekante für jede Bogensekunde von 0°0′ bis 1°20′*. Third ed. Bonn, Dümmler, 1946, vi, 293 p. 19 × 26 cm.

This is a photographic reprint of a corrected first edition, Berlin and Bonn, Dümmler, but some errors still remain. A corrected second edition appeared in 1939. The first Russian edition came out at Moscow, in 1937, and the second Russian edition in 1938. For other details see MTE 135, elsewhere in this issue.

523[D, R].—RUDOLF BOSSHARDT, *Tafeln zum Abstecken von Kreisbogen nach Polarkoordinaten*, Stuttgart, Konrad Wittwer, 1940, iv, 106 p. 12.5 × 17.7 cm.

Consider a circular arc ABC with center O. Let the tangents at A and C meet in T, and let OT meet the arc in B, and the chord of the arc CA in E; also let $OA = r$, $\angle COA = \alpha$.

Then (1) $AT = CT = r \tan \frac{1}{2}\alpha$; (2) arc $ABC = r\pi\alpha$ (sexagesimal)/180; (3) $= r\pi\alpha$ (centesimal)/200; (4) $BT = r[\sec(\frac{1}{2}\alpha) - 1]$; (5) $AE = CE = r \sin \frac{1}{2}\alpha$; (6) $EB = r(1 - \cos\frac{1}{2}\alpha)$. For $r = 100$, Table I gives 3D values of (1), (2), (4), (5), (6), each with Δ, for α(cent. degrees) $= 1^g(1^g)120^g(0^g.5)180^g$; and (2), (5), (6), each with Δ, for $\alpha = 180^g(1^g)200^g$.

Let $AC = s$, $\angle BAT = \psi$, and arc $ABC = b$; then with varying parameter range b, and $r = 12(1)50(2)100(5)200(10)500(20)1000(50)1500(100)5000$, Table II gives ψ(cent.) and ψ(sexag.), each to the nearest second; also $b - s$ (2D) with Δ(3D). For $r = 12 - 17$, $b = 1,\ 2(2)24(1)45$; for $r = 105$–155, $b = 1(5)70(2)100(5)140$; and for $r = 200$–5000, $b = 1,\ 10(10)160$.

Table III is for changing from centesimal to sexagesimal division, and conversely. The remaining pages are devoted to uses of the tables with surveying instruments and to a 23-item Literature-List.

The author expresses the view that he has made a new contribution to surveying tabular material.

R. C. A.

524[E].—NBSCL, *Tables of the Exponential Function e^x*. Second edition (211 copies). Washington, D. C., U. S. Govt. Printing Office, 1947. [xviii], 549 p. 20.7 × 27 cm. For sale by the Superintendent of Documents, Washington, D. C. $3.00. On the back of the volume MT2 is printed; presumably MT1 was *Table of the First Ten Powers of the Integers from 1 to 1,000*.

This volume of which the first edition appeared in 1939 (see *MTAC*, v. 1, p. 438), was the first of the extensive volumes which this computing group prepared. We have already noted that it has long been out of print (*MTAC*, v. 3, p. 65), and that six slips in it had been found (*MTAC*, v. 1, p. 161, 198; v. 2, p. 314, 352). All of these have been corrected in the new edition which is appreciably thinner than the old edition, on account of thinner, but good, paper having been used in its composition. The binding is very substantial. On the whole the printed pages of the table are not quite as black in the review copy as in the copy of the first edition now before me.

The number of pages in the volume is the same as formerly. The new title-page has been considerably changed. The old material of p. [vii–viii] has been eliminated, and a brief new "Preface to the second edition" by Dr. LOWAN, chief of the Computation Laboratory, appears on p. [vii] of the new edition. Our total of 549 pages differs from the 535 on the last table page of the volume through the fact that title-pages preceding Tables II–VIII are not numbered. The title-page for Table I was counted in [xviii]. The increase of price of this excellent volume from $2.00 to $3.00 still leaves it an extraordinarily cheap product for these days, and will probably do little to retard its further rapid sale.

R. C. A.

525[E].—JØRGEN RYBNER, *Tabeller til Brug ved Løsning af Ligningen $tgh(b + ja) = r\underline{/}\phi$*. Copenhagen, Jul. Gjellerups Forlag, 1943, 8 p. 17 × 25 cm. Offset print.

These tables are of

$$f(r) = 2r/(1 + r^2), \qquad g(r) = 2r/(1 - r^2),$$

for $r = [0(.001).999; 6D]$, Δ.

If $r > 1$, take $1/r$ and $f(1/r) = f(r)$, $g(1/r) = -g(r)$.

If $\tanh(b + ia) = r(\cos\phi + i \sin\phi)$, $\tanh 2b = f(r) \cos\phi$, $\tan 2a = g(r) \sin\phi$. Compare the 16 nomograms for $\tanh(b + ia)$ in J. RYBNER, *Nomogrammer over komplekse hyperbolske Funktioner*, 1947 (RMT 526).

526[E, H].—JØRGEN RYBNER, *Nomogrammer over komplekse hyperbolske Funktioner. Nomograms of Complex Hyperbolic Functions.* Copenhagen, Jul. Gjellerups Forlag, 1947, 36 p. text + 54 leaves printed on one side with nomogram material + plastic rule (3 × 31.1 cm.) with hair line. 21.3 × 31 cm.

The individual investigator with limited facilities at his disposal has always found the evaluation of a hyperbolic function of a complex variable to be a tedious and time consuming chore. Professor Rybner has taken great care to produce an accurate set of steel engraved nomograms covering the sinh, cosh and tanh (designated tgh) which should be very useful to such a worker who has a large number of cases to consider.

There are 13 separate sheets to cover the cosh with an accuracy approaching three figures. The cosh is expressed in Cartesian form by the equation

$$\cosh (b + ia) = p + iq.$$

The values of b, which in electrical work corresponds to attenuation, is expressed in both nepers and decibels, while a, which corresponds to phase shift, is given in degrees and radians, each divided on a decimal basis. Each nomogram carries the formulae for computation and the rules concerning sign and periodicity.

The sinh function also covers 13 sheets and is treated similarly.

The tanh function covers 16 sheets and is given in the polar form

$$\tanh (b + ia) = r\underline{/\theta}.$$

Again b is expressed in both nepers and decibels but a is given in radians only; θ is expressed in degrees and minutes or degrees and decimals. Suitable formulae for computation, for sign and periodicity are included on each sheet. The reason for expressing tanh in polar form rather than in the rectangular form of the sinh and cosh is not given. The polar form however is convenient for the transmission engineer who frequently wants to compute short circuit or open circuit impedances.

The text, which is given in both Danish and English, includes a number of formulae for circular and hyperbolic functions and also some formulae for four-terminal networks and transmission lines. Approximation formulae which appear to be new are given for very small values of b.

In addition to the above, certain nomograms useful to the engineer are also included. These are concerned with

1. Conversion from rectangular to polar form i.e., $x + iy = r\underline{/\theta}$.
2. Conversion from $R\underline{/a}$ to $(1 + r/\theta)$.
3. Reflection loss and reflection phase shift.
4. Resonance circuits and filters.

The ring binding at the top of the nomograms is somewhat less convenient than conventional binding at the side of a sheet.

The general usefulness of the nomograms would be enhanced by typical examples.

A number of errors in notation were observed in a somewhat cursory turning of the pages, as follows:

p. 32, l. −5, *for* B_2, B_2 *read* B_1, B_2.

sheet 10 for tanh, at the lower right, *for* $b = 0.75 - 0.5$, *read* $b = 0.25 - 0.5$.

sheets 1, 3, 6, 9, 10, 11, 12, 13, 14, 15, 16 at the top:

$$\textit{for } \operatorname{arctg} \frac{\sin 2a}{\sinh 2a}, \qquad \textit{read } \operatorname{arctg} \frac{\sin 2a}{\sinh 2b}.$$

The need for the tanh nomograms was realized many years ago by transmission engineers. Aside from Professor KENNELLY'S extensive and well-known *Atlas*,[1] which uses r and θ as the basic coordinate system, the only other published set of charts known to the writer is that of Professor HENRY E. HARTIG, "Charts for transmission line problems," *Physics*, v. 1, 1931, p. 380–387, which interchanges the coordinates and contours of the *Atlas*. Pro-

fessor Hartig's charts use a neper-quadrant coordinate system with $R/\underline{\theta}$ contours for sinh and cosh. For the tanh the contours are $R/\underline{\theta}$ in the 1st, 4th, 5th, etc. octants and $R^{-1}/\underline{\theta}$ contours in the 2nd, 3rd, 6th, 7th, etc. octants. With these charts the answer, as a function of length, always lies on a straight line. They also have the virtue of clarifying the periodic nature of the functions.

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[1] A. E. KENNELLY, *Chart Atlas of Complex Hyperbolic and Circular Functions*. Third ed. rev. and enl., Cambridge, Mass., 1924, 66 p. Kennelly published also *Tables of Complex Hyperbolic and Circular Functions*. Second ed. rev. and enl., Cambridge, 1921. vi, 240 p.

527[E, S].—PARRY MOON, A. "A table of Planck's function from 3500 to 8000°K," *Jn. Math. Phys.*, v. 16, 1937, p. 133–157. 17.5 × 25.4 cm. Also M. I. T., E. E. Dept., *Contribution* no. 131, 1938. B. *A Table of Planck's Function 2000 to 3500°K*, Cambridge, Mass., Mass. Inst. Techn., Dept. Electr. Engineering, 1947, 80 p. 15.2 × 22.8 cm. C. "A table of Planckian radiation," Opt. Soc. Amer., *Jn.*, v. 38, 1948, p. 291–294.

The growing use, in science and engineering, of Planck's equation for the radiation from a blackbody emphasizes the need for more satisfactory tabulated values of this function. In photometry and radiometry, stress is being laid to an increasing extent upon calculations based on blackbody radiation. In colorimetry a similar movement is in progress—a movement that has been particularly marked since 1931, when the Commission Internationale de l'Éclairage standard distribution coefficients were introduced. A number of tables of Planck's function are available, but all have disadvantages when values are to be obtained with the minimum of calculation.

Planck's equation is

$$J(\lambda) = C_1\lambda^{-5}[e^{C_2/\lambda T} - 1]^{-1}, \tag{1}$$

where λ = wavelength (micron), T = absolute temperature (°K), $C_1 = 36970$, $C_2 = 14320$; C_1 and C_2 are in accordance with the international temperature scale;[2] $J(\lambda)$ = spectral radiosity of a blackbody (watts cm.$^{-2}$ micron^{-1}) at wavelength λ.

If $x = \lambda T/C_2$, $yx^5 = J(\lambda)\lambda^5/C_1$, equation (1) becomes

$$yx^5 = [e^{1/x} - 1]^{-1}. \tag{2}$$

If $C'_2 = .4342944819 \cdot C_2 = 6219.096981$,

$$J(\lambda) = 36970\lambda^{-5}[10^{6219.096981/\lambda T} - 1]^{-1}. \tag{3}$$

Actual computations were made from this equation. T.I. in A is for C'_2/T, for $T = 2000$°K-(10°)3500°(100°)10000°(1000°)20000°K, T.2 is of C_1/λ^5, for $\lambda = .26(.01)1(.1)3(1)8$; T.3 is a double entry table for λ and T, of $J(\lambda)$, for

$$\lambda = [.26(.01).75(.05)1(.1)3;\ 5S],\ \Delta, \text{ and } T = 3500(100°)8000°\text{K}.$$

It is believed that in general the values of this table are correct to the last figure, though because of rounding process there may be an error in a few instances of as much as 5 in the sixth figure.

In B the table is of $J(\lambda)$ for $\lambda = [.38(.01).76;\ 8S]$, Δ, and $T = 2000(10°)3500$°K. The values were computed to 10 digits by means of electrically-driven computing machines. The values were checked by use of fifth differences, and were then rounded off to the 8 figures given in the table.

This new table should be particularly useful, since it embraces the region of filament temperatures of incandescent lamps, and the two tables extend from spectral distributions approximating the radiation from a carbon-filament lamp to those approximating the radiation from an overcast sky.

The tables in A and B were calculated under Mr. Moon's direction by more than a score of computers. In using (3) the powers of 10 were read directly from Peters' *Zehnstellige Logarithmen*, v. 1.

Tables based on the relation (2) have been given by JAHNKE & EMDE,[1] FREHAFER & SNOW,[2] FABRY,[3] YAMAUTI & OKAMATU.[4] Since there is only one independent variable, the table can be condensed into comparatively small space while still covering the entire range of wavelength and temperature. These tables, however, are anything but satisfactory. Interpolations are nearly always necessary; and the labor entailed in the interpolations, as well as in the computation of x from the interpolated values of λ and T and the computation of $J(\lambda)$ from the interpolated value of y, is considerable. In fact, comparative tests showed that $J(\lambda)$ could be computed directly from (1) more easily than it could be obtained from Yamauti & Okamatu tables,[4] a logarithm table and a modern calculating machine being used in both cases.

Present needs require a table of double entry, with the two independent variables λ and T. Tables of this kind have been computed by FORSYTHE,[5] FOWLE,[6] FREHAFER & SNOW,[7] and SKOGLAND.[8] These tables are based on the values of either 14330 or 14350 for C_2, instead of the international standard value[9] of 14320. The change to 14320 is easily made with existing tables, but requires additional computation which tends to reduce the serviceability of the tables. The Skogland results are being used extensively, both in this country and abroad, but they cover only a limited range of temperature and are confined to the visible spectrum. The Fowle table and Frehafer & Snow table are not subject to these restrictions in temperature and wavelength but the values are in general calculated to only 2 or 3 significant figures.

Extracts from text

EDITORIAL NOTE: The radiation formula of MAX K. E. L. PLANCK (1858–1947) was first published in "Ueber eine Verbesserung der Wienschen Spektralgleichung," Deutsche phys. Gesell., *Verhandlungen*, v. 2, 1900, p. 202. This note is reprinted in the *Ostwald's Klassiker d. exakten Wissenschaften*, v. 206, 1923, p. 53–55. In the bibliography of the function a reference may be given also to NBSCL, *Miscellaneous Physical Tables. Planck's Radiation Functions . . .*, 1941.

[1] E. JAHNKE & F. EMDE, *Tables of Functions*, second ed., Leipzig, 1933, p. 45–47 [Also in later editions.—EDITOR]

[2] MABEL K. FREHAFER & C. SNOW, *Tables and Graphs for Facilitating Computations of Spectral Energy Distribution*. Nat. Bureau of Standards, *Misc. Publs.*, no. 56, 1925, T. I.

[3] C. FABRY, *Introduction générale à la Photométrie*, Paris, 1927.

[4] Z. YAMAUTI & M. OKAMATU, "Tables for Planck's radiation formula," Tokyo, Electrotechn. Lab., *Researches*, no. 395, 1936, part I, 39 p. (English abstract); no. 402, 1937, part II, 51 p. in English.

[5] W. E. FORSYTHE, "1919 report of standards committee on pyrometry," Optical Soc. Amer., *Jn.*, v. 4, 1920, p. 331.

[6] F. E. FOWLE, "Radiation from a perfect (blackbody) radiator," in Nat. Res. Council, *Intern. Critical Tables*, v. 5, 1929, p. 238–243.

[7] MABEL K. FREHAFER & C. SNOW, *idem*, T.II.

[8] J. F. SKOGLAND, *Tables of Spectral Energy Distribution*. Nat. Bureau of Standards, *Misc. Publs.*, no. 86, 1929.

[9] G. K. BURGESS, "The international temperature scale," Nat. Bureau of Standards, *Jn. Research*, v. 1, 1928, p. 637.

528[F].—N. G. W. H. BEEGER, "Second extension of the table of least exponents ξ for which $2^\xi \equiv 1 (\mathrm{mod}\ p)$," *Nieuw Archief voor Wiskunde*, s. 2, v. 22, p. 310–311, 1948. 15.7 × 23.6 cm.

This is an extension for the range $309672 < p < 320000$ of the exactly similar table reviewed in RMT 279, v. 2, p. 71. The existence of the manuscript of this table is there announced. The author calls attention to the fact that $p = 318781$ has the small exponent 828 so that p divides $2^{414} + 1$. Incidentally so does $p = 853669$.

D. H. L.

529[F].—M. KRAÏTCHIK, *Théorie des Nombres*, v. 3; *Analyse Diophantine et Applications aux cuboïdes rationnels*. Paris, Gauthier-Villars, 1947, xi, 135 p. 15.8 × 24.5 cm.

This work contains the following four tables:

T.I (p. 36–55). This table is to facilitate the solution of the diophantine equation

$$x^4 + \alpha x^2 y^2 + y^4 = u^2$$

by giving solutions of the associated congruence

$$y^4 + \alpha y^2 + 1 \equiv u^2 \pmod{p}.$$

The trivial cases of $\alpha = \pm 2$ are omitted. The moduli considered are the 24 odd primes <100.

T.II (p. 83–85). This is a list of rational integral cuboids, that is, a list of triples (x, y, z) of positive integers such that the sum of the squares of any two of them is a perfect square. Thus x, y, and z are the edges of a rectangular parallelopiped whose faces have integral diagonals. The author considers only those cases in which x, y, z are not all even. This implies that exactly one of them is odd and this odd one is taken as z. The list is arranged according to z and extends to $z \leq 10^4$. There are 284 of these cuboids of which only 39 are primitive, that is, x, y, z have no common factor.

T.III (p. 112–113). This table is wrongly labeled. The title should read $x^2 + 16y^2 \equiv z^2 \pmod{p}$. The second column should be headed z^2, not x^2. The moduli considered are the primes ≤ 43 except 2, 3, 7. The table gives the values of x^2 or $y^2 \pmod{p}$ for each possible given value of $z^2 \not\equiv 0 \pmod{p}$.

T.IV (p. 122–131). This is a list of the 241 primitive cuboids with an odd value of $z < 10^3$. For each such z the values of x, y, z are given with three pairs of generators. See also *MTAC*, v. 2, p. 167.

D. H. L.

530[F].—D. B. LAHIRI, "On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$. Part II." Calcutta Math. Soc., *Bull.*, v. 39, 1947, p. 33–52. 18.7 × 24.2 cm. For a review of Part I see *MTAC*, v. 3, p. 23, RMT 459.

This Part contains a large table of 171 congruence relations between Ramanujan's function $\tau(n)$, defined by

$$\Delta(x) = \sum_{n=1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}$$

and the sums of powers of the divisors of n, the moduli being certain divisors of $2^{12}3^65^3 \cdot 7 \cdot 11 \cdot 13 \cdot 691$. The following is a typical example.

$$9504\tau(n) \equiv 13650\sigma_{11}(n) - 691[11(6n-5)\sigma_9(n) - 5\sigma_1(n)] \pmod{2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 691}.$$

The general form is

$$A\tau(n) \equiv \sum_{k=0}^{k} P_k(n)\sigma_{2k+1}(n) \pmod{M}$$

where A and the polynomial P depend on M. The large number of apparently different relations is due in great measure to the numerous congruence relations existing between the σ's. In many cases, and in the above example, the constant A is not prime to the modulus M so that one obtains a congruence for $\tau(n)$ itself only for a certain divisor of M. This happens for instance whenever M is divisible by 2^{11}, 3^6, 11, or 13. The highest powers of the various primes for which there is a congruence relation for $\tau(n)$ itself are 2^{10}, 3^5, 5^2, 7, and 691. There is also an interesting table giving the 41 products of the functions $\Phi_{r,s}(x)$, defined in RMT 459, which are expressible linearly in terms of other Φ's and the generator Δ. It is from this table that the main table was produced.

D. H. L.

531[H].—JOHN T. PETTIT, "A speedy solution of the cubic," *Mathematics Mag.*, v. 21, Nov.–Dec. 1947, p. 94–98. 17.1 × 25.4 cm.

In solving $y^3 + py + q = 0$, if p and q have the same signs, change the signs of the roots and consider $y^3 \pm |p|y \mp |q| = 0$. With the substitution $y = |q|z/|p|$, transform the equation to $z^3/(1 - z) = \pm |p|^3/|q|^2 = K$. For the general cubic, $x^3 + bx^2 + cx + d = 0$, reduce to the previous case by taking $x = y - \frac{1}{3}b$, $p = c - \frac{1}{3}b^2$, $q = d - \frac{1}{3}bc + 2b^3/27$.

The relation $z^3/(1 - z) = K$ is represented by a graph on p. 96, and tables are given on p. 97–98 for $z = -3(.01) + 1.5$ and K to 3S, $-\infty < K < \infty$. Over this range z is a single-valued function of K, and one real root is thus obtained. On reducing the general equation to a quadratic the other roots are found to be $x = [q/2p][z \pm (-4K - 3z^2)^{\frac{1}{2}}] - \frac{1}{3}b$.

The procedure and the table printed in this article are essentially those given in more detail by H. A. NOGRADY, *A New Method for the Solution of Cubic Equations*, 1936; see *MTAC*, v. 1, p. 441f. Other references on the numerical solution of the cubic are given in *MTAC*, v. 2, p. 28f.

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NOTE BY S. A. J.: The Pettit table is obviously wholly unreliable, since there are from 2 to 78 units of error in the last-figure values of K, for the following 14 values of z: −2.22, −1.29, −1.22, −1.19, −1.04, −1.02, −0.53, −0.52, −0.51, −0.23, −0.22, +0.34, 0.52, 1.18.

532[H, I, L].—H. S. CARSLAW & J. C. JAEGER, *Conduction of Heat in Solids*. Oxford, Clarendon Press, 1947, vi, 386 p. 15.5 × 24 cm. Miss MARTHA E. CLARKE is credited with assistance in computation of the tables.

Appendix II: "The error function and related functions," p. 370–373. There are 4D tables of $e^{x^2}\operatorname{erfc} x$, $4\pi^{-\frac{1}{2}}xe^{-x^2}$, $2\pi^{-\frac{1}{2}}e^{-x^2}$, $\operatorname{erf} x$, $\operatorname{erfc} x$, $2i\operatorname{erfc} x$, $4i^2\operatorname{erfc} x$, $6i^3\operatorname{erfc} x$, $8i^4\operatorname{erfc} x$, $10i^5\operatorname{erfc} x$, $12i^6\operatorname{erfc} x$, for $x = 0(.05)1(.1)3$.

$$\operatorname{erf} x = 2\pi^{-\frac{1}{2}}\int_0^x e^{-t^2}dt;\ \tfrac{1}{2}\pi^{\frac{1}{2}}\operatorname{erfc} x = \int_x^\infty e^{-t^2}dt;\ i^n\operatorname{erfc} x = \int_x^\infty i^{n-1}\operatorname{erfc} t\,dt,$$

$n = 1, 2, \cdots$, $i^0\operatorname{erfc} x = \operatorname{erfc} x$.

Appendix IV: "The roots of certain transcendental equations," p. 377–379. T.1, the first six roots, α_n, to 4D, of $\alpha\tan\alpha = C$, for $C = .001, .002(.002).01, .02(.02).1(.1)1(.5)2(1)10(5)20(10)60(20)100, \infty$. The roots of this equation are all real if $C > 0$. T.2, the first six roots, α_n, to 4D, of $\alpha\cot\alpha + C = 0$, for $C = -1(.005) - .99(.01) - .9(.05) - .8(.1) + 1(.5)2(1)10(5)20(10)60(20)100, \infty$. The roots of this equation are all real if $C > -1$. T.3, the first six roots, α_n, to 4D, of $\alpha J_1(\alpha) - CJ_0(\alpha) = 0$, for $C = 0(.01).02(.02).1(.05).2(.1)1(.5)2(1)10(5)20(10)60(20)100, \infty$. T.4, the first five roots, α_n, to 4D, of $J_0(\alpha)Y_0(k\alpha) - Y_0(\alpha)J_0(k\alpha) = 0$, for $k = 1.2, 1.5(.5)4$.

Appendix V: "Table of Laplace transforms $\theta(p) = \int_0^\infty e^{-pt}v(t)dt$," p. 380–381. There are twenty-eight entries under $\theta(p)$, $v(t)$.

Extracts from text

EDITORIAL NOTE: A table of the first seven roots of $\alpha\tan\alpha = C$ was given by A. B. NEWMAN & L. GREEN, "The temperature history and rate of heat loss of an electrically heated slab," Electrochem. Soc., *Trans.*, v. 66, 1934, p. 355, for $C = [0, .1, .5, 1(1)10, \infty$; 4D]. For various solutions of $\alpha\cot\alpha + C = 0$, or $\tan\alpha = -\alpha/C$, see *MTAC*, v. 1, p. 203, 336, 459; v. 2, p. 95. In this connection reference may also be given to J. C. JAEGER & MARTHA CLARKE, "Numerical results for some problems on conduction of heat in slabs with various surface conditions," *Phil. Mag.*, s. 7, v. 38, 1947, p. 507.

533[K].—Bell Telephone Laboratories, *Table of Individual and Cumulative Terms of the Point Binomial* $(q + p)^n$. (Inspection Engineering Memoranda, no. IEM-4; Quality Assurance Dept. 2530.) June 4, 1947, 398 leaves. Hectographed on one side only. 21.5 × 27.7 cm. This edition is not available for general distribution but the volume is to be published by the Van Nostrand Company in 1949.

Consider N independent trials with a constant probability P of success. The probability of a total of exactly X successes is given by

$$b(X, N; P) = \binom{N}{X} P^X (1 - P)^{N-X}, \qquad 0 < P < 1.$$

This is the binomial distribution or, as it is sometimes called, the point binomial. The summatory function

$$B(X, N; P) = b(0, N; P) + b(1, N; P) + \cdots + b(X, N; P)$$

gives the probability of at most X successes. Both functions are of fundamental importance in all applications of probability theory. For a numerical evaluation of $B(X, N; P)$ tables of the beta function can be used, but this procedure is rather cumbersome. It is therefore customary to use the two classical approximations to $b(X, N; P)$ and $B(X, N; P)$. Putting

$$z = \{X + \tfrac{1}{2} - NP\}\{NP(1-P)\}^{-\frac{1}{2}}$$

the Laplace limit theorem states that as $N \to \infty$

$$B(X, N; P) \to 2\pi^{-\frac{1}{2}} \int_0^z e^{-\frac{1}{2}t^2} dt.$$

This approximation is accurate only if NP is very large. If N is large but $NP = a$ is of moderate magnitude, the Poisson approximation

$$b(X, N; P) \approx e^{-a} a^X / X!$$

is much better. For moderate values of N neither approximation is satisfactory. More accurate asymptotic expansions are available, but are rather cumbersome for routine use. In modern sampling, in industrial quality control, and many other applications we frequently deal with comparatively small values of N. In such cases one has to resort to tedious computations or to unsafe approximations. Therefore the present tables will be greeted with a feeling of relief by many practical statisticians: they solve the problem completely at least within the range $50 \leqslant N \leqslant 100$.

It should be noticed that $b(X, N; P) = b(N - X, N; 1 - P)$ and $B(X, N; P) = 1 - B(N - X - 1, N; 1 - P)$. Accordingly there is no need to tabulate the values for $P > \frac{1}{2}$. The present tables contain 11 independent sections, each covering one value of N in the range 50(5)100. In every case $b(X, N; P)$ and $B(X, N; P)$ are tabulated in adjacent columns for $P = [0(.01).50; 6D]$. The number of X-entries is usually considerably smaller than N since only values of X near NP correspond to $b(X, N; P)$ values significantly different from zero. The last digit is not absolutely reliable. It appears to the reviewer that the accuracy is sufficient for most practical applications, and that the tabular interval in P is sufficiently small. It is to be hoped that these useful tables[1] will be extended beyond $N = 100$.

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[1] The following acknowledgment is in the introduction: Thanks are extended to Mr. E. G. Andrews for his cooperation and assistance in computing the table; to Dr. H. Nyquist, Dr. John Riordan, and Dr. H. W. Bode for their aid in arranging for the project; to Dr. F. L. Alt and his able assistant, Miss Bettie Boyd, for setting up the project on the Computing System; to Miss Alice G. Loe for work on the printer, for checking all phases of the work, and for computing some 5% of the values appearing in the table which could not be completed by the Computing System because of limited time; and to other members of the Quality Assurance Department for guidance and cooperation.

Harry J. Romig

534[K].—C. V. L. CHARLIER, *Elements of Mathematical Statistics, also L. v. Bortkiewicz, Table of Poisson's Frequency Function edited and translated by* J. ARTHUR GREENWOOD. Cambridge, Mass., 1947, ii, 120 p. Orders to be placed with J. A. Greenwood, 25 Winthrop St., Brooklyn 25, N. Y. $3.00. Plastic binding. 17.5 × 22.7 cm.

The first edition of this little work of CARL VILHELM LUDVIG CHARLIER (1862–1934) appeared in 1910 as *Grunddragen af den matematiska Statistiken*, Lund. The next edition was by Charlier in German, *Vorlesungen über die Grundzüge der mathematischen Statistik*, Lund, 1920. 127 p. 17.5 × 22 cm. This differed from the Swedish edition by the addition of a fifteenth chapter and an appendix of the following five tables: **T.I**, $Q(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dx$, for $x = [-2.9(.1) - 2(.01) + 2(.1)2.9; 4D]$; **T.II**, $x = R(u)$, inverse of Q for $u = [0(.01).99; 4D]$; **T.III**, $\phi_0 = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$, for $x = [0(.01)3(.1)4.9; 4D]$; **T.IV**, $\phi_3(x) = d^3\phi_0/dx^3 = (-x^3 + 3x)\phi_0$; **T.V**, $\phi_4(x) = d^4\phi_0/dx^4 = (x^4 - 6x^2 + 3)\phi_0$, Hermite polynomials ϕ_3 and ϕ_4 for $x = [0(.01)3(.1)4.9; 4D]$.

There are various small tables scattered throughout the work. In the English edition these are numbered 1–40 and the tables of the Appendix, p. 101–116, are numbered 41–44. **T.41** of $-x = -R(u)$, is **T.II**, above, rearranged. We are told that eight last-figure errors were corrected by comparison with the 10D table of T. KONDO & E. M. ELDERTON, *Biometrika*, v. 22, p. 368–370; also K. PEARSON, *Tables for Statisticians and Biometricians*, v. 2, 1932, p. 2–10.

T.42 is of $\phi_{-1} = Q(x)$, ϕ_0, $-\phi_2$, ϕ_3, $-\phi_5$, for $x = [0(.01)4; 7D]$; ϕ_4 and $-\phi_6$ are also given, for the same range, to 4D. ϕ_{-1} and ϕ_0 are taken from those by W. F. SHEPPARD in *Biometrika*, v. 2, 1903, p. 182–188; also in PEARSON, *Tables for Statisticians and Biometricians*, v. 1, Cambridge, 1914, p. 2–7. Mr. Greenwood states that ϕ_2, ϕ_3 and ϕ_5 were computed, partly to 13D, partly to 10D, using the NBSCL, *Tables of Probability Functions*, v. 2, 1942, and the tables of N. R. JØRGENSEN, *Undersøgelser over Frequensflader og Korrelation*, Copenhagen, 1916, and then rounded to 7D. The Greenwood values of ϕ_2, ϕ_3, and ϕ_5 are identical with those of Jørgensen, whose 7D table of ϕ_6 is rounded off to a 4D table. ϕ_4 is the same as Charlier's 4D **T.V.**

T.43 is the table referred to in the title of the book under review, L. v. BORTKIEWICZ, *Das Gesetz der kleinen Zahlen*, Leipzig, 1898, p. 49–52, giving the 4D values of $m^x e^{-m}/x!$ for $m = .1(.1)10$, $x = 0(1)24$. (Compare *MTAC*, v. 1, p. 19.)

"**T.44** was newly computed by the translator," it is stated. It gives "the limits within which the Skewness and Excess must lie in order that a frequency curve of type A may yield positive frequencies." There are only 18 entries $E = 0(.03).48, .50$, and for these the corresponding 3D values of $|S| <$ are given.

Such are the principal tables in the latest form of an elementary work first published nearly 40 years ago.

R. C. A.

535[L].—ENZO CAMBI, *Eleven- and Fifteen-Place Tables of Bessel Functions of the First Kind, to All Significant Orders*. New York, Dover Publications, 1948, vi, 154 p. 22 × 29 cm. $3.95.

This new addition to the ever expanding list of tables of the Bessel functions consists essentially of two tables of the functions and two additional tables of the coefficients used in their computation. The author explains his purpose in this new work by the following statement in the preface: "A great scientific need was met just ten years ago, when the British Association for the Advancement of Science published its volume of ten place tables of *Bessel Functions . . . of Orders Zero and Unity*, for x varying from 0 to 16 at interval 0.001 and from 16 to 25 at interval 0.01. In many war time problems, however, there was a constant demand for tables of the values of such functions to higher integral orders, especially for Bessel functions of the first kind. The present volume offers a contribution in this field."

In Part I of the work, 143 pages, the essential tables are as follows:

Table I gives the values of $J_n(x)$ for $x = [0(.01)10.5; 11D]$ and the order n is extended to values for which J_{n+1} becomes smaller than $.5 \times 10^{-11}$ for the entire range of x. Actually a few values of J_{30} are included.

Table II includes the same functions as those in Table I except now the range is $x = [.001(.001).5; 15D]$ and is extended to include values of J_n which are greater than 10^{-15}. The table thus terminates with a few values of J_{11}.

In Part II, p. 144–148, we find the values of the coefficients of the Taylor's series in h for $x = 2(1)6$ and $n = 2(2)16$ to 12D. Through $n = 6$ the expansions include terms to h^{11} and through $n = 16$ they terminate with h^{10}.

In Part III, p. 149–153, there are recorded the values of A_m in the expansion,

$$J_n(x + h) = A_0J_0(h) + A_1J_1(h) + A_2J_2(h) + A_3J_3(h) + \cdots,$$

for $x = 7(1)10$ and $n = 2(2)30$. Through $n = 16$ these are to 12D, but "for the higher values of the order $n(n > 16)$, the coefficients are limited to the figures required for the computation of J_n to 12–13 places."

In a fourth part, p. 154, the author gives a brief bibliography of Bessel functions.

In computing the tables described above, the author began with the well-known tables of MEISSEL for J_0 and J_1. The values of $J_n(x)$, $n > 1$ were computed up to $x = 1.5$, by the power series, from 1.5 to 6.5 by means of Taylors' series, and from 6.5 to 10.5 through the addition formula, for even n. The recurrence formula for Bessel functions was used to compute the values for odd n, except in the cases $n = 3$ and $n = 5$, where the formula might have introduced errors beyond the accuracy limit of the table.

In checking the accuracy of the 11th place error in Table I, the following formulae were used:

$$1 = J_0 + 2J_2 + 2J_4 + \cdots,$$
$$\sin x = 2J_1 - 2J_3 + 2J_5 - 2J_7 + \cdots.$$

In computing Table II, J_0 was interpolated from the 16D table of HAYASHI and the values of the other functions of even order were computed from the power series and the use of interpolation formulae. The values of the functions of odd order were found by means of the recurrence formula: $J_n(x) = (x/2n)[J_{n+1}(x) + J_{n-1}(x)]$, which could be used safely since x was small.

The present table seems to fill a useful place in spite of the fact that the Computation Laboratory of Harvard University is providing more extensive tables of some of the same functions. This project is providing the values of $J_n(x)$ for $x = 0(.001)25(.01)99.99$, $n = 0(1)3$ to 18D and $n = 4(1)14$ to 10D; then for $n = 15(1)100$, $x = [0(.01)99.99; 10D]$. The first six volumes for $n = 0(1)15$ appeared in 1947. But this basic work because of its size and cost will not be available readily to every computer who well may wish to use Bessel functions. But the compactness and low cost of the 11D and 15D work under review will recommend it to everyone who may desire to include an extensive set of values of the Bessel functions in his library.

H. T. D.

EDITORIAL NOTE: Dr. CAMBI requests that the following corrections be made in his volume of tables reviewed above: p. v, l. −3, and p. 149, l. 1, *for* Taylor Series, *read* Additional Series; p. 47, $J_3(6.93)$, *for* .35385 88467 2, *read* .35385 88367 2.

536[L].—GEORGE A. CAMPBELL & RONALD M. FOSTER, *Fourier Integrals for Practical Applications*, New York, Van Nostrand, 1948. 178 p. 15.8 × 23.7 cm. $3.50.

The response of a linear oscillating system to a harmonic signal, $a \cos pt + b \sin pt$, is in general harmonic, $A \cos pt + B \sin pt$; and the behavior of such a system can be regarded as known if its response to a harmonic signal of arbitrary frequency and phase has been determined. The response to an arbitrary signal can then be obtained in three steps: (i)

decomposition, or analysis, of the signal into its harmonic components, (ii) finding the response for each of the harmonic components, and (iii) superposition, or synthesis, of the component responses. If the incoming signal is periodic, of period T, its decomposition takes the form of a Fourier series,

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T},$$

and the synthesis of the component responses amounts to the summation of a Fourier series. For non-periodic signals, for instance for so-called transients, the frequency spectrum is continuous in general, and the decomposition takes the form of a *Fourier integral*

$$G(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} F(f) df. \tag{1}$$

Given the signal, $G(t)$, its analysis into harmonic components involves finding $F(f)$ which is given by

$$F(f) = \int_{-\infty}^{\infty} e^{-2\pi i f g} G(g) dg. \tag{2}$$

Again, if the harmonic components, with amplitudes $F(f)$, of the response are known, their synthesis is based on (1). Thus in the case of Fourier integrals harmonic analysis and synthesis use the same type of integral, and the practical importance of an extensive table of integrals of this type is apparent.

Fourier integrals have many other applications, for instance to the solution of certain types of differential and integral equations (Cf. E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937). Again, the so-called Heaviside, or operational, calculus is based on Laplace integrals which are essentially one-sided Fourier integrals, that is Fourier integrals corresponding to functions $G(t)$ which vanish for negative t.

The book under review contains what is generally acknowledged to be the most extensive published list of Fourier integrals. It was originally published, under the same title, in 1931 as Monograph B-584 of the *Bell Telephone System Technical Publications*, reprinted in the *Collected papers of George A. Campbell*, New York, American Telephone and Telegraph Company, 1937; the original edition of 1931 was reproduced photographically, with minor corrections, in 1942. Though the present book does not mention any of the previous editions, it appears to be a photographic reproduction of the 1942 edition. No attempt seems to have been made at bringing the material up to date, even though the authors are known to have continued collecting valuable material after 1931.

There are 33 pages of explanatory text. An introduction explains the importance of Fourier integrals, the relationship between $F(f)$ and $G(g)$, the terminology, the role of the unit impulse and its relationship to the harmonic function cis $(2\pi ft) = e^{2\pi i f t}$, the use of the tables for obtaining Fourier integrals. Next the general processes for deriving pairs of functions F and G ("mates"), and elementary transformations as well as various properties of such pairs are explained. There follow brief sections on mates based on the normal error law, essentially singular pairs, such as $F = (2\pi if)^n$ and its mate, pairs for which F and G are in a given relation, for instance $F(x) = G(x)$, on Fourier series, on the use of contour integrals, and on practical applications of the tables. There is also a summary of the descriptive part, and a list of notations and abbreviations used throughout the book.

Table I contains 763 pairs of mates F, G under the following 13 headings: General processes for deriving the mate; Elementary combinations and transformations; Key pairs; Rational algebraic functions of f; Irrational algebraic functions of f; Exponential and trigonometric functions of f or f^{-1}; Exponential and trigonometric functions of f^2; Other elementary transcendental functions of f; Other transcendental functions of f; Fourier series given as pairs; Contour integrals. Paths parallel real axis; Contour integrals. Closed paths; Contour integrals. Paths with arbitrary end-points.

Table II contains admittances of and transients in physical systems, illustrating the application of Fourier integrals to transient problems. There are 39 physical systems listed under 3 headings: Time variable; Space variable; Two space variables. In each case a brief

description of the physical system is given together with its admittance (whose knowledge is equivalent to the knowledge of the response to an arbitrary harmonic signal), its response to an instantaneous unit impulse, to a unit step (a signal of constant intensity which starts at $t = 0$) and to a harmonic signal which arrives at $t = 0$.

A few valuable features of this carefully compiled book deserve special praise. They are: A carefully devised and scrupulously maintained system of notation which enables the authors to present the results very concisely and yet quite precisely (for instance v, w always stand for integers, m, n for non-negative, and j, k, l for positive integers). "Key pairs," formulae containing numerous parameters and thereby parent formulae for many special and limiting cases listed in the various special sections; such key pairs are collected separately. Numerous cross references. Very helpful special explanations, given in specially devised symbols, referring to alternative expressions, related transform pairs, special cases in which the restrictions may be relaxed, and similar matter. The only criticism which this reviewer has to level against this excellent work is—that there is not enough of it, i.e. that it has not been brought up to date.

A. ERDÉLYI

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537[L].—GUSTAV DOETSCH (1892–), *Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch.* (*Die Grundlehren der math. Wissen. in Einzeldarstellungen*, v. 54.) Berlin and Göttingen, Springer, 1947, x, 186 p. Offset print. 16 × 24.2 cm.

The first part of the book (p. 1–71) contains a condensed but clear account, without proofs, of the important properties of the Laplace transform and a description of how it can be used to solve differential equations. Specific illustrations of the various possibilities are given, including a heat conduction problem where a blind application of the complex inversion formula leads to an incorrect result. The author's notation is

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt = \mathfrak{L}\{F(t)\}.$$

The second part consists of a table of inverse transforms: each entry gives $f(s)$, the corresponding $F(t)$, and the abscissa of convergence. This is the arrangement adopted in the shorter tables of M. F. GARDNER & J. L. BARNES, *Transients in Linear Systems studied by the Laplace Transformation*, New York, Wiley, 1942, v. 1, p. 334–356, and R. V. CHURCHILL, *Modern Operational Mathematics in Engineering*, New York, McGraw-Hill, 1944, p. 295–302 (122 entries in the table), and appears to be more convenient for applications than the opposite arrangement as used, for example, by N. W. MCLACHLAN & P. HUMBERT, *Formulaire pour le Calcul Symbolique* (*Mémorial des Sciences Mathématiques*, no. 100), Paris, Gauthier-Villars, 1941. In addition, pages 164–185 of Doetsch contain an index of functions, arranged alphabetically by the symbols used to denote the functions, with the definition of each function and references to all the formulae where it appears. This index enables the table to be used with some facility as a table of direct transforms.

This is the most extensive published table of Laplace transforms. Its scope can be seen from the following summary: Operations (56 formulae); Rational functions (65); Irrational functions (105); Logarithmic, inverse trigonometric functions (90); Exponential functions (102); Trigonometric and hyperbolic functions (74); Gamma and related functions (37); Integral functions (sine, cosine, exponential integrals, etc.) (42); Confluent hypergeometric functions (45); Bessel functions (108); Spherical harmonics (14); Elliptic integrals (28); Theta functions (15); Other special functions (12). The tables of J. COSSAR & A. ERDÉLYI, *Dictionary of Laplace Transforms*, 1944–1946 (*MTAC*, v. 1, p. 424–425, v. 2, p. 76, 215–216), not yet made available to the general public, contain more formulae for transcendental functions, but fewer for elementary functions.

The book is reproduced from typescript, with the formulae very clearly written by hand.

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EDITORIAL NOTE: In effect, LAPLACE introduced in 1779 the transformation which has received his name by showing the correspondence between the two functional domains which it relates [P. S. LAPLACE, "Sur les suites," *Mém. Acad Sci.*, 1779; *Oeuvres Complètes*, v. 10, p. 1–89]. The Laplace transformation may be viewed as a limiting form of a transformation used by Euler [E. G. C. POOLE, *Introd. to the Theory of Linear Differential Eqns.*, Oxford, 1936, p. 137]. In this sense the transformation is related to work which preceded Laplace's. The definite integral form of the direct Laplace transformation was applied by Laplace to the solution of differential and difference equations in 1782 [*Oeuvres*, v. 10, p. 209–338]. Laplace's book on the theory of probability, 1812, illustrates the many uses to which he puts the transformation. This Note is adapted from GARDNER & BARNES, *l.c.*, p. 360.

There is also a table (112 entries) in ERNST HAMEISTER, *Laplace Transformation. Eine Einführung für Physiker, Elektro-Machinen und Bauingenieure.* Munich and Berlin, 1943; also Ann Arbor, Mich., Edwards Bros., 1946, 147 p.

538[L].—HARRY J. GRAY, RICHARD MERWIN, & J. G. BRAINERD, *Solutions of the Mathieu Equation*, Amer. Inst. Electr. Engineers, *Technical Paper* 48–70. 1948, 17 p. 20.5 × 27.8 cm. Offset print. To nonmembers by mail, 55 cents.

The equation whose solutions are tabulated is

$$d^2y/dt^2 + \epsilon(1 + k\cos t)y = 0. \tag{1}$$

If $2z$ be written for t, a for 4ϵ, and $-q$ for $2\epsilon k$, the resulting equation is

$$d^2y/dz^2 + (a - 2q\cos 2z)y = 0, \tag{2}$$

this form being preferable in analytical work. The equation has four types of solution depending upon the values of (a, q):

1°. The first solution is bounded and periodic in π or 2π, the second being non-periodic and tending to $\pm\infty$ as $z \to +\infty$.
2°. Both solutions are bounded and periodic with period $2s\pi$, s integral $\geqslant 2$.
3°. Both solutions are bounded, but non-periodic.
4°. Both solutions are non-periodic, but one tends to $\pm\infty$ while the other tends to zero as $z \to +\infty$.

The values of (a, q) pertaining to 1° lie on curves symmetrical about the a axis, and these divide the (a, q)-plane into regions[1] where the solutions have the forms in 2°–4°. For 2°, 3°, the region is called stable, and for 4° unstable, by virtue of the type of solution. The paper under review is concerned with solutions of the type 2°–4°, which occur in the following physical problems:

i. Frequency modulation in radio telephony, and the acoustical warble tone;
ii. Pendulum whose support executes harmonic motion in a vertical plane;
iii. Oscillations of side-rod electric locomotives;
iv. Sub-harmonics in mechanical vibrating systems, and in conical loud-speaker diaphragms;
v. Pseudo-rectilinear motion of loud-speaker coil in non-uniform radial magnetic field;
vi. Stability of structural columns subject to a longitudinal harmonic force;
vii. Unsymmetrical short-circuits on water-wheel generators under capacitive loading.

From this incomplete list, it is evident that solutions of Mathieu's equation for certain ranges of a, q are needed, and the question arises as to the most suitable way in which the data should be presented.

The two solutions of (1) are taken in the form

$$g(t) = u_1\cos\mu t - u_2\sin\mu t,$$

and

$$H(t) = u_1\sin\mu t + u_2\cos\mu t = Mh(t);$$

u_1 (even), u_2 (odd), being functions with period π or 2π,

$$\cos 2\pi\mu = 2g(\pi)h'(\pi) - 1,$$

and

$$M = [-g(\pi)g'(\pi)/h(\pi)h'(\pi)]^{\frac{1}{2}}.$$

The initial conditions are

$$g(0) = h'(0) = 1, \qquad g'(0) = h(0) = 0.$$

The quantities tabulated are: (a) $g(t)$, $h(t)$, for $t = 0(.1)3.1, \pi$; $\epsilon = 1(1)10$; and $k = 0(.1)1$; the total number of entries for *each* function is 3300; (b) μ, M for $\epsilon = 1(1)10$, and $k = 0(.1)1$, the total number of entries for *each* parameter[2] being 100. M is real, imaginary or complex, but it is preferable to have real solutions. In all cases the data are given to 5D.[3] Interpolation will usually not yield data comparable in accuracy with that tabulated. The interval 1 for ϵ precludes data being obtained for intermediate points. In the reviewer's opinion it is preferable to have the solution in functional rather than in numerical form, especially in cases where it is non-periodic. A clearer picture of the physical behavior of a system may then be visualized. Such solutions were discussed by Dr. GERTRUDE BLANCH, in *MTAC*, v. 2, p. 263–266, and examples have been given by the writer.[1,4] By having either a comprehensive table of a, $a^{\frac{1}{2}}$, q, μ, or a large scale chart (two or three feet square), and using a form of solution such that μ is *never* complex, the coefficients in the series solution may be found quickly. The functional form of the solution then follows immediately. For instance if in (2) we take $a = 1$, $q = .16$, then $\mu = \simeq .08$. Apart from a constant multiplier, using t for z, the two solutions are

$$y_1(t) = e^{.08t}[\cos t - .021 \cos 3t + \cdots + .94 \sin t - .0175 \sin 3t + \cdots], \tag{3}$$

$$y_2(t) = y_1(-t). \tag{4}$$

If these refer to a physical system, its behavior with increase in t can be visualized immediately, whereas from purely numerical solutions this would not be so. (3), (4) are neither odd nor even. This is advantageous, since odd and even solutions (like those tabulated for (a)) would *both* tend to $\pm\infty$ as $t \to +\infty$, whereas $y_1(t) \to \pm\infty$, while $y_2(t) \to 0$.

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[1] See N. W. McLACHLAN, *Theory and Applications of Mathieu Functions*. Oxford, 1947, figures 8, 11.

[2] The values of μ are half those corresponding to equation (2).

[3] The authors make the following statement: "The tables are given to five decimal places, but because they are extracted from larger ones and no rounding has been done, the digit in the fifth decimal place may be in error, and it seems best to state the accuracy of the tables as four decimal places. In this connection, it may be pointed out that the tables were obtained on the ENIAC, using 10-digit numbers and an interval $\Delta t = .0004$ in the corresponding difference equation. Rounding and truncation errors cut the accuracy so that 5- or 6-figure accuracy might be expected."

[4] N. W. McLACHLAN, "Computation of the solutions of $(1 + 2\epsilon \cos 2z)y'' + \theta y = 0$; frequency modulation functions," *Jn. Appl. Physics*, v. 18, 1947, p. 723–731.

539[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 9, *Tables of the Bessel Functions of the First Kind of Orders Sixteen through Twenty-Seven*. By the Staff of the Laboratory, Professor H. H. AIKEN, technical director. Cambridge, Mass., Harvard University Press, 1948, x, 764 p. This is the first v. of the Bessel function series to be paged. 19.5 × 26.7 cm. $10.00. Offset print. Compare *MTAC*, v. 2, p. 176f, 261f, 344; v. 3, p. 102, 117–118.

This is the seventh, and largest so far published, of the thirteen planned volumes in the monumental edition of tables of Bessel functions prepared at Harvard University by means of the IBM Automatic Sequence Controlled Calculator. It contains tables of $J_n(x)$, $n = 16(1)27$, $x = [0(.01)99.99; 10D]$. For $n = 0(1)15$, the interval up to 25 had been .001. Since many of the early values of the 12 functions are zero, the first cases of those with the

significant value .00000 00001 are respectively as follows: $J_{16}(3.12)$, $J_{17}(3.60)$, $J_{18}(4.10)$, $J_{19}(4.62)$, $J_{20}(5.16)$, $J_{21}(5.72)$, $J_{22}(6.29)$, $J_{23}(6.88)$, $J_{24}(7.48)$, $J_{25}(8.09)$, $J_{26}(8.71)$, $J_{27}(9.35)$.

Among all entries in the table, 112 725, there are only 296 previously published to at least 10D; there are 288 given by MEISSEL (1895) for $x = [0(1)24;\ 18D]$, 1 given by MEISSEL (1891), $J_{20}(20)$ to 20D, and 7 given by HAYASHI (1930) for $x = 1, 2, 10(10)50$, to at least 15D. Since the Harvard volume was published, however, the tables by CAMBI have appeared (reviewed elsewhere in this issue, RMT **537**). The ranges of values here partially duplicating those of the Harvard volume are $x = [0(.01)10.5;\ 11D]$, $n = 16(1)27$.

The tapes used in computing the tables of the volume under review were coded by JOHN A. HARR, who also supervised the operation of the ASCC and the preparation of the manuscript.

R. C. A.

540[L].—P. I. KUZNETSEV, "Funktsii Lommelia ot Dvukh Mnimykh Argumentov" [Lommel functions with two imaginary arguments], *Prikladnaia Matem. i Mekhanika*, v. 11, Oct. 1947, p. 555–560.

$$Y_n(w, z) = i^{-n}U_n(iw, iz),\quad \Theta_n(w, z) = i^{-n}V_n(iw, iz),$$

$$U_n(w, z) = \sum_{m=0}^{\infty} (-1)^m (w/z)^{n+2m} J_{n+2m}(z),\quad V_n(w, z) = \sum_{m=0}^{\infty} (-1)^m (z/w)^{n+2m} J_{n+2m}(z),$$

$$Y_n(w, z) = \sum_{m=0}^{\infty} (w/z)^{n+2m} I_{n+2m}(z),\quad \Theta_n(w, z) = \sum_{m=0}^{\infty} (z/w)^{n+2m} I_{n+2m}(z).$$

There are 6D tables of $Y_n(w, z)$, $\Theta_n(w, z)$, for $n = 1, 2$, $w = 1(1)6$, $z = 1(1)6$; with graphs. There are also 4D tables, and graphs, of $V(x, t)$, $10^3 J(x, t)$, for $t = 6.56$ and ∞, $x = 0(100)400$; and for $x = 400$, $t = 6, 10(5)25$.

541[L].—W. LANE & D. SWEENEY, *Table of Legendre Polynomials P_n (cos θ) for $n = 0(1)20$, and $\theta = 0(1°)180°$ to Six Decimals*. United States Atomic Commission, Oak Ridge, Tennessee, Los Alamos Scientific Laboratory MDDC–780, LADC–361. Date of Manuscript Feb. 12, 1947; document declassified March 18, 1947. 10 p. 20.4 × 26.8 cm. Printed.

The contents of this publication are indicated by its title. The first sentence of the text is "The accompanying table of Legendre Polynomials was computed for use because no tables of the functions of the desired range were available from other sources." A correct statement would have been the following: "What we have tried to do is to compute again a small section of a 6D table readily available in many libraries, namely: $P_n(\cos\theta)$, $n = 1(1)32$, $\theta = 0(10')90°$, by A. Ḥ. H. TALLQVIST, Finska Vetenskaps-Societeten, *Acta*, s. 2, v. 2A, no. 11, 1938. We did not look at this table."

At the end of the first paragraph of the text is the following statement: "The table is accurate to six decimal places for values of $n < 15$, and five decimal places for values of $n \geqslant 15$." Here are suggestions as to how much dependence can be placed on this statement: A comparison of Tallqvist's table, mentioned above, was made with the 220 entries of Lane & Sweeney's table for $n = 1(1)20$, $\theta = 0(1°)10°$. Tallqvist's 10D table for $n = 1(1)8$, $\theta = 0(1°)90°$, Finska Vetenskaps-Societeten, *Acta*, v. 33, no. 4, 1905, and A. LODGE's 7D table for $n = 1(1)20$, $\theta = 0(5°)90°$, R. Soc. London, *Trans.*, v. 203A, 1904, p. 100–101, were used for further checking. For the 154 entries $n < 15$ said by Lane & Sweeney to be "accurate to six decimal places" there are 104 errors of from 1 to 11 units in the sixth decimal place; there are 34 unit errors and 35 cases of errors of more than 3 units. Next, let us consider the statement of the authors regarding the 66 entries for $n \geqslant 15$. Here there are 35 errors, 5 of them being errors of 2 units in the fifth decimal place.

Such are illustrations of ignorance and incompetence in a Government Laboratory at an important center.

R. C. A.

542[L].—NBSCL, *Tables of Bessel Functions of Fractional Order*, Volume 1. New York, Columbia University Press, 1948, xlii, 413 p. 19.7 × 26.5 cm. $7.50.

This is the first of two volumes dealing respectively with the tabulation of the functions $J_\nu(x)$ and $I_\nu(x)$, prepared under the able direction of Dr. LOWAN. The main part of this volume is occupied (p. 1–271) with the following tables of $J_\nu(x)$: $\nu = -\frac{2}{3}$ and $-\frac{1}{3}$, $x = [0(.001).9(.01)25;\ 10D]$; $\nu = -\frac{1}{2}$ and $-\frac{1}{4}$, $x = [0(.001).8(.01)25;\ 10D]$; $\nu = \frac{1}{4}$ and $\frac{1}{3}$, $x = [0(.001).6(.01)25;\ 10D]$; $\nu = \frac{2}{3}$ and $\frac{3}{4}$, $x = [0(.001).5(.01)25;\ 10D]$. δ^2, or modified δ^2, are provided throughout; and also δ^4, or modified δ^4, for $x = .05(.001).15$. Furthermore, for $x < .05$, tables for $x^{-\nu}J_\nu(x)$, with their second central differences, are also provided, where interpolation close to the origin is not feasible.

The interval in argument has been so chosen that interpolation may yield the maximum attainable accuracy over most of the range covered. Everett's interpolation coefficients E_2, F_2, E_4 and F_4 are tabulated at interval .001. In the range for fourth central differences indicated above, it is always possible to obtain an accuracy of at least 7S, by means of δ^2.

When x is large, use is made of the auxiliary tables (p. 273–383) of $A_\nu(x)$ and $B_\nu(x)$, for $x = [25(.1)50(1)500(10)5000(100)10000(200)30000;\ 10D]$; $J_\nu(x) = A_\nu(x)\cos(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi) - B_\nu(x)\sin(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi)$. There are 10D values of $\sin x$ and $\cos x$ (p. 386–387) for $x = 0(.01)1.6,\ 2(1)40$, and on p. xlii are 15D values of various constants.

The values of the first 30 zeros of $J_\nu(x)$, to 10D, and for $\pm\nu = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$, are given on p. 384–385. These appeared earlier (for the first 8 in more extended form) in *MTAC*, v. 1, p. 353–354, 1945; and v. 2, p. 118–119, 1946.

In the long Introduction is a note by Mr. H. E. Salzer on "Bessel functions considered as functions of their order." There are also (p. xxxiii–xli) "Note on modified second differences for use with Everett's Interpolation formula" and "Bibliography" (30 selected titles).

Bessel functions of fractional orders greater than unity may be obtained from the present tables by employing recurrence relations.

The preliminary manuscript of $J_\nu(x)$ to 13D or 13S, first prepared and subjected to a sixth difference test, has been previously referred to in *MTAC*, v. 1, p. 93, 300. No earlier published table gave more than 7D values of $J_\nu(x)$. Hence, even in this respect the volume under review marks a very notable contribution to tabular material in varied fields where such functions are of use, for example: wave propagation in stratified media with a constant gradient in the index of refraction,[1] asymptotic solutions of ordinary linear differential equations, quantum mechanics, and engineering problems.

R. C. A.

[1] R. E. LANGER, "On the connection formulas and the solutions of the wave equation," *Physical Rev.*, v. 51, 1937, p. 669–676.

543[L].—NBSCL, *Tables of the Bessel Functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leq x \leq 1$.* (*Applied Mathematics Series*, no. 1.) Washington, D. C., 1948, x, 60 p. 20 × 26 cm. For sale by the Superintendent of Documents, Washington. 35 cents.

These tables inaugurate the Applied Mathematics Series of the Computation Laboratory of the Bureau of Standards; see *MTAC*, v. 3, p. 65. In a foreword A. M. WEINBERG & E. P. WIGNER state that the tables were required in connection with the construction of huge nuclear chain reacting piles at Hanford, Washington.

The table gives (p. 3–31, 35–60) the values of $Y_0(x)$, $Y_1(x)$, for $x = 0(.0001).05(.001)1$, and the values of $K_0(x)$, $K_1(x)$, for $x = 0(.0001).033(.001)1$. A. N. Lowan states in the Introduction that "With the exception of a few entries close to the origin, the entries of the present table were obtained by interpolation in the BAAS tables. . . ." In general the functions are given to 8S or 8D except near the origin where a few values are given to 9S. First and second differences are provided.

Further to simplify the problem of interpolation in $Y_0(x)$ and $Y_1(x)$ for x small, the auxiliary functions $C_0(x)$, $C_1(x)$, $D_0(x)$, and $D_1(x)$ as defined in the British Association volume are given (p. 2) for $x = [0(.0001).005;\ 8D]$, Δ. Similar auxiliary functions of $K_0(x)$ and $K_1(x)$ are given (p. 34) for $x = 0(.001).03$. The Introduction states that the eight-figure values may be in error by as much as two units, and the nine-figure values may be in error by as much as four units of the last place retained. Known errors occur in $\Delta Y_1(0.0096)$ and $\Delta K_1(0.700)$, see MTE **133**. The tables have been spot checked by J. A. HARR, and several pages have been differenced by the reviewer. No other errors were found.

H. H. AIKEN

EDITORIAL NOTE: The tables of $K_0(x)$, $K_1(x)$ here printed were earlier listed as manuscripts in *MTAC*, v. 1, p. 165, 300.

544[L, S, T].—G. W. KING, "The asymmetric rotor. VI. Calculation of higher energy levels by means of the correspondence principle," *Jn. Chem. Phys.*, v. 15, Nov. 1947, p. 820–830. 19.7 × 26.8 cm.

This is the final printed record of an investigation, of which a duplicated account, embodied in two bimonthly progress reports, has been reviewed at some length in RMT **467** (*MTAC*, v. 3, p. 27–29). Table I (p. 825) is identical with Table I of the August report; Table II (p. 827) with Table II (June); and Table III (p. 829) with Table III (August).

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545[M].—G. R. MACLANE, "Table of integrals," p. 369–370 of H. M. JAMES, N. B. NICHOLS & R. S. PHILLIPS, *Theory of Servomechanisms*. (M.I.T. *Radiation Laboratory Series*, v. 25.) New York, McGraw-Hill, 1947.

The table is of integrals of the type

$$I_n = (2\pi i)^{-1} \int_{-\infty}^{\infty} g_n(x)dx/[h_n(x)h_n(-x)]$$

where

$$h_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$
$$g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots + b_{n-1},$$

and the roots of $h_n(x)$ all lie in the upper half plane. The table lists the integrals $I_n(x)$, for $n = 1(1)7$. $I_1 = b_0/2a_0a_1$; $I_2 = (-a_2b_0 + a_0b_1)/2a_0a_1a_2$; etc.

546[M. S].—W. G. POLLARD & R. D. PRESENT, *On Gaseous Self-Diffusion in Long Capillary Tubes*. United States Atomic Energy Commission, Columbia University, MDDC–1521. Document declassified Dec. 11, 1947. 29 p. 20.4 × 26.8 cm. Offset print.

Appendix IV, "The evaluation of the integral Q," p. 28–29,

$$Q = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \sin^2 t \cos^2 t\, dt \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos x\, e^{-2a \cos t \cos(x/\lambda)} dx = \pi/16 - \tfrac{1}{2}\pi K,$$

where

$$K = \int_0^{\frac{1}{2}\pi} \sin^2 t \cos^2 t\, [\mathbf{H}_1(2ai/\lambda \sin t) - iJ_1(2ai/\lambda \sin t)]dt.$$

The following five values of K: .0560, .0834, .1066, .1186, and .1250, are given as corresponding respectively to $a/\lambda = .25, .5, 1, 2$, and ∞.

547[Q].—JOHN B. IRWIN, "Tables facilitating the least-squares solution of an eclipsing binary light-curve." *Astrophysical Jn.*, v. 106, 1947, p. 380–426. The numbering of equations in this review continues the numbering in RMT 549.

As is well known, the fractional light l of an eclipsing binary system can, during eclipse, be written as

$$l = 1 - fL_1 \tag{6}$$

where L_1 denotes the fractional luminosity of the eclipsed component and $f = f^x(k, p) = f(\delta, r_1, r_2, x)$ is a function defined by equations (1)–(5). Moreover, the apparent separation δ of the centers of both components can be expressed as $\delta^2 = \sin^2\theta \sin^2 i + \cos^2 i$, where θ denotes the true anomaly of the eclipsing component in its relative orbit and i, the inclination of the orbital plane to the celestial sphere. Hence, in general, $f = f(\theta, r_1, r_2, i, x)$, where θ is a known function of the time and r_1, r_2, i, x are constants to be determined by orbital analysis. Suppose that this has been done by some approximate method and that Δl represents the residual difference between light observed at any particular moment and that computed on the basis of the approximate elements. These residuals are due partly to the observational errors, and partly to possible inaccuracies in the adopted system of the elements. The corrections ΔL_1, Δr_1, Δr_2, Δi, and Δx required to minimize the sum of squares of the residuals can, however, be obtained as follows. Differentiating equation (6) above we find that

$$\Delta l = -f\Delta L_1 - L_1\left\{\frac{\partial f}{\partial r_1}\Delta r_1 + \frac{\partial f}{\partial r_2}\Delta r_2 + \frac{\partial f}{\partial i}\Delta i + \frac{\partial f}{\partial x}\Delta x\right\}. \tag{7}$$

The residual Δl of each observation of light during eclipse supplies us with an equation of condition of this form for determining the requisite corrections. Provided that these residuals are small and that their number exceeds sufficiently the number of the unknowns, a least-squares solution of such a set of equations then yields the most probable values of the required corrections as well as their errors.

The whole difficulty in application of this process centers around the evaluation of the four partial derivatives of f with respect to r_1, r_2, i, and x for every observed point. The derivative with respect to x presents the least difficulty; for a differentiation of equation (1) yields readily

$$\partial f/\partial x = 6(f^D - f^0)/(3 - x)^2. \tag{8}$$

The remaining three partial derivatives can be obtained by differentiating the right-hand sides of equations (2)–(5) behind the integral sign. In doing so we find that

$$\begin{cases} \dfrac{\partial f^0}{\partial r_2} = 2\dfrac{r_2}{r_1^3} I^0_{-1,0} & \dfrac{\partial f^D}{\partial r_2} = 3\dfrac{r_2^2}{r_1^3}\sqrt{\dfrac{\delta}{r_2}}\, I^1_{-1,1} \\[2ex] \dfrac{\partial f^0}{\partial \delta} = -2\dfrac{r_2}{r_1^3} I^1_{-1,0} & \dfrac{\partial f^D}{\partial \delta} = -3\dfrac{r_2^2}{r_1^3}\sqrt{\dfrac{\delta}{r_2}}\, I^1_{-1,1} \end{cases} \tag{9}$$

where

$$\pi r_2^{\beta+\gamma+m+1} I^m_{\beta,\gamma} = \int_{\delta-r_2}^{c} [r_2^2 - (\delta - x)^2]^{\frac{1}{2}\beta}[2(s - x)]^{\frac{1}{2}\gamma}[\delta - x]^m dx$$

and $c = s$ or $\delta + r_2$ depending on whether the eclipse is partial or annular. Since, moreover, f is a homogeneous function of r_1, r_2, and δ(i.e., it depends, not on r_1, r_2, δ directly, but only on their ratios), Euler's theorem on homogeneous functions can be invoked to prove that

$$\frac{\partial f}{\partial r_1} = -\frac{r_2}{r_1}\frac{\partial f}{\partial r_2} - \frac{\delta}{r_1}\frac{\partial f}{\partial \delta} \tag{10}$$

while ultimately

$$\frac{\partial f}{\partial i} = -\sin 2i \frac{\partial f}{\partial \cos^2 i} = -\frac{\sin i \cos i \cos^2\theta}{\delta}\frac{\partial f}{\partial \delta}. \tag{11}$$

Five-decimal tables of the $I^{m}_{k,\gamma}$-integrals recently constructed by the reviewer for another purpose (see RMT 515), can be used to facilitate the computation of the required derivatives by means of equations (8)–(11) above. This approach using closed formulae is, however, apt to be fairly laborious—chiefly because the I-integrals have not been tabulated in terms of the customarily known parameters k and p, but rather in terms of a certain auxiliary angle α which has nothing to do with the present problem. An alternative way for evaluating the requisite partial derivatives for any pair of k and p suggests itself: namely a numerical differentiation of Tsesevich's tables.[1] The necessary formulae by which this can be done were outlined by the reviewer some time ago[2] and need not be reproduced here. The accuracy which can be reached in this way is naturally limited by the accuracy of Tsesevich's tables; whereas a five-digit accuracy is easily attainable by means of the literal formulae (8)–(11) and Kopal's tables of the I-integrals, a numerical differentiation of Tsesevich's tables cannot yield more than three correct figures in partial derivatives. This appears, however, to be all that is needed for the improvement of preliminary elements of eclipsing binary systems by way of differential corrections at present and probably for some time to come.

Computers of photometric orbits of eclipsing variables have recently been put under a deep debt of obligation to Professor Irwin for having completed the extensive set of bivariate tables of all partial derivatives of equation (7) in terms of k and p (or q for annular eclipses), which are the subject of the present review. They were evaluated, not by means of the literal formulae (8)–(11), but by a numerical differentiation of Tsesevich's tables, although—Irwin states—the literal formulae and Kopal's tables were used to compute all derivatives for the $k = .975$ columns, the $p = -.975$ rows, and many values of the $p = -1$ row as well as numerous spot-checks. The extent of Tsesevich's tables limited the accuracy of Irwin's tables to three significant figures. The arguments of tabulation are $k = 0(.1).8(.05)1$, $p = -1(.05) - .8(.1).8(.05)1$ if the eclipse is an occultation ($r_1 = r_a$), and $k = .2(.1).8(.05)1$, $p = -1(.05) - .8(.1).8(.05)1$ during a partial transit while, during annular phase, $q = .1(.1)1$. Except for tables 4, 8, 12, 16, 20, and 24, additional columns are inserted for $k = .975$ and $p = -.975$ to facilitate interpolation.

Irwin's paper contains altogether 32 bivariate tables which can be divided broadly in the following two groups:

I. Occultation 3D Tables (smaller star eclipsed; $r_1 = r_a$)

	$x = 0$	.4	.6	1
$r_a \dfrac{\partial f^s}{\partial r_b}$:	1A	9	17	1B
$r_a \dfrac{\partial f^s}{\partial r_a}$:	2A	10	18	2B
$\dfrac{-r_a r_b}{\cos^2\theta} \dfrac{\partial f^s}{\partial \cos^2 i}$	3A	11	19	3B

II. Transit 3D Tables (larger star eclipsed; $r_1 = r_b$)

	$x = 0$	.4	.6	1
$\dfrac{r_b}{k} \dfrac{\partial f^s}{\partial r_b}$:	5A	13	21	5B
$\dfrac{r_b}{k} \dfrac{\partial f^s}{\partial r_a}$:	6A	14	22	6B
$\dfrac{-r_b}{k \cos^2\theta} \dfrac{\partial f^s}{\partial \cos^2 i}$	3A	15	23	7

In addition, Tables 4 and 8 list 4D values of $\frac{1}{4}(3 - x)^3\partial f^*/\partial x$ for the case of an occultation and a transit, respectively. Tables 12 and 20 contain the occultation values of $\partial f^*/\partial x$ for $x = .4$ and .6, respectively, whereas tables 16 and 24 contain the corresponding transit values.

In the past several months the present reviewer has made extensive use of Irwin's tables (communicated to him in advance of publication) and had also a frequent opportunity for their spot-checking on the basis of the exact formulae. The tables were found to be correct to one unit of the last place in every case under examination. No tabular differences are given, but in most tables the first differences turn out to be small enough to permit a linear interpolation in both variables. There is no doubt that the publication of Irwin's tables will greatly facilitate the computation of accurate photometric orbits of eclipsing binary systems and, in this way, contribute significantly to further development of this rapidly growing subject. The only criticism which can be raised concerns some of Irwin's notations. Although these are adequately explained in the introduction preceding the tables, some of them are needlessly complicated (too many superscripts), and many depart rather drastically from common usage without sufficient reason. This fact, the reviewer feels, is bound to make the reading of Irwin's memoir inadvertently more difficult to a general reader than it need have been.

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[1] Leningrad, Astr. Inst., *Bûull.*, nos. 45 (1939) and 50 (1940), see RMT 548.
[2] Amer. Philos. Soc., *Proc.*, v. 86, 1943, p. 342.

548[Q].—V. P. T̂SESEVICH, "Tablit̂sy dlîa opredeleniîa elementov orbit zatmennykh zvezd tipa Algolîa̋" [Tables for the determination of the orbital elements of eclipsing stars of the Algol type], no. **45**, 1939, p. 115–152. "Tablit̂sy fotometricheskikh faz zatmenîĭ peremennykh zvezd tipa Algolîa̋" [Tables of the photometric phases of eclipses of variable stars of the Algol type], no. **50**, 1940, p. 283–366. Leningrad, Astron. Institut, *Bûull.*

The determination of orbital elements of eclipsing binary systems represents an important task of theoretical astronomy, calling for an extensive use of functions that can be practically dealt with only in tabular form. Some tables which have been constructed for this purpose represent rather remarkable feats of table construction and are, moreover, of a sufficiently general nature to warrant a comprehensive review.

The basic functions of double star astronomy whose numerical values are needed in tabular forms can be described as follows. Let r_a, $r_b (r_a \leqslant r_b)$ denote the radii of two luminous circular discs and let δ be the distance between their centers. If, at any moment, $\delta < r_a + r_b$, one disc will eclipse the other and a loss of light will result, which depends on the geometrical circumstances as well as on the distribution of brightness $J(r)$ over the disc undergoing eclipse. This latter function is usually approximated by a cosine law of the form $J(r) = J_0(1 - x + x\cos\gamma)$, where J_0 denotes the central brightness of the eclipsed disc, x is the "coefficient of darkening," and γ is such that $\sin\gamma = r$, the fractional distance from the center.

Analytical formulae expressing the resultant loss of light can be set up and evaluated in a closed form for any kind of eclipse.[1] It turns out that, if the disc undergoing eclipse appears uniformly bright ($x = 0$), the respective loss of light (proportional to the eclipsed area) can be expressed in terms of circular and algebraic functions; but if $x \neq 0$, we encounter elliptic integrals, not only of the first and second kinds, but also of the third kind which belong, fortunately, to the "circular" class and are therefore expressible in terms of incomplete elliptic integrals of the first and second kinds with complementary moduli. Even so, however, their explicit forms are so complicated as to render them of little practical use unless their numerical values are available in tabular forms.

The function f which represents the loss of light arising from an eclipse can, in general, be made to depend on two parameters formed by any ratio of r_a, r_b, and δ. For the purpose

of table construction it is customary to adopt, as such parameters, the ratio of the radii $k = r_a/r_b$ and the geometrical depth of the eclipse $p = (\delta - r_b)/r_a$ chosen so that, by definition, $0 \leqslant k \leqslant 1$, while p is constrained to vary from 1 at the moment of first contact to -1 at the moment of internal tangency—regardless of whether the latter marks the beginning of totality or of annular phase. Furthermore, the loss of light f is, for practical reasons, customarily normalized to vary between 0 and 1 through dividing it by the respective loss of light at the moment of internal tangency, and so normalized values of the fractional loss of light are denoted as $\alpha^x(k, p)$. For any given pair of k and p the numerical value of $\alpha^x(k, p)$ depends, naturally, also on the degree of darkening x of the disc undergoing eclipse. In practice, however, it is sufficient to evaluate the α's for "uniform" ($x = 0$) and "completely darkened" ($x = 1$) discs only, because the α's appropriate for any intermediate value of x follow from the interpolation formula

$$(1) \qquad f^x(k, p) = [(3 - 3x)/(3 - x)]f^0(k, p) + [2x/(3 - x)]f^D(k, p).$$

It should be added that, because of the way in which the f's have been normalized, their numerical values depend also on whether the disc undergoing eclipse is the smaller or the larger of the two (i.e. whether the eclipse is an occultation or a transit). If the eclipsed disc appears uniformly bright, the ratio of the fractional losses of light during an occultation and transit for any given pair of k and p is equal to k^2; in the presence of a limb-darkening this ratio turns out, however, to be less simple and therefore separate tables of $\alpha^D(k, p)$ must be constructed depending on whether the eclipse is an occultation (α') or a transit (α'').

An inversion of $\alpha(k, p)$ yields the geometrical depth of the eclipse $p(k, \alpha)$ as a function of the ratio of the radii k and the fractional loss of light α. Unlike $\alpha(k, p)$, the inverse function $p(k, \alpha)$ defies, however, any explicit analytical formulation so that its values are obtainable only by numerical inverse interpolation. The values of appropriate to intermediate degrees of darkening cannot, moreover, be obtained by linear interpolation between the two extreme cases, but for each x the inversion has to be carried out anew.

Owing to their importance in the computation of orbits of eclipsing binary stars, several tabulations of the fractional loss of light $\alpha(k, p)$ and of the geometrical depth of the eclipse $p(k, \alpha)$ have been attempted in the past decades. The first of such tabulations, to 3D, of $p(k, \alpha)$ by H. N. Russell[2] (*Astroph. Jn.*, v. 35, 1912, p. 315; T.I on p. 333) and of $p(k, \alpha')$ and $p(k, \alpha'')$ by H. N. Russell & H. Shapley (Tables Ix and Iy in *Astroph. Jn.*, v. 36, 1912, p. 243 and 390), are mentioned here for historical reasons only; for, these tables were both too small and insufficiently accurate to be of permanent value.[3]

In 1931, new tables of the α- and p-functions appropriate for eclipses of uniformly bright discs were completed by Wend [4] and Hetzer [5] as dissertations under the guidance of the late Professor Julius Bauschinger. Wend constructed a 4S table of α as a function of $k = .3(.01)1$ and δ/r_b (not p). Since there is no common upper (or lower) bound on δ/r_b during the eclipses, the range of his second parameter varies from table to table, though the increment is always equal to .01. While Wend completed a table essentially equivalent to $\alpha(k, p)$, Hetzer tabulated a quantity $W^k = 1 + kp(k, \alpha)$, to 4D, for $k = 0(.01)1$ and $\alpha = 0(.01)1$.

In 1938 and 1939 Ferrari published [6] 4D tables of the fractional loss of light, relevant to completely darkened discs, for both occultation and transit and covering partial as well as annular eclipses. His paper (i) contains a table of $\alpha'(k, p)$ for $k = 0(.05)1$ and $p = 1(-.01)-1$, while paper (ii) contains a table of $\alpha''(k, p)$, the range of the arguments being $k = .4(.05)1$ and $p = 1(-.01)-1$ for the partial (transit) eclipse, and $-1(-.01)-2.5$ for the annular eclipse; first tabular differences (k-wise) are also printed. An additional table gives $\alpha''(k, p)$, during annular eclipse, for $k = .3(.05).4$ and $p = -1(-.05)-3.25$.

The most extensive and accurate existing set of published α- and p-tables so far, however, are those (45, 50) of Tsesevich under review. No. 45 is devoted to tables appropriate to the case of complete darkening ($x = 1$). After a brief introduction, defining the function to be tabulated and describing the methods of their evaluation, we find T.I containing 5D values of $\alpha'(k, p)$ tabulated for $k = 0(.05)1$ and $p = 1(-.01)-1$, followed by a 4S table of

its inverse function $p(k, \alpha')$ tabulated at the same intervals in k and for $\alpha' = 0(.01)1$ (T.II). T.III then gives 5D values of $\alpha''(k, p)$ for $k = .2(.05)1$ and $p = 1(-.01)-1$, followed again by its inverse $p(k, \alpha'')$ tabulated to 4D for $k = .2(.05)1$ and $\alpha'' = 0(.01)1$ (T.IV).

No. 50 opens (apart from certain auxiliary tables) with a 5S table of $\alpha(k, p)$ appropriate for uniformly bright discs and tabulated for $k = 0(.05)1$ and $p = 1(-.01)-1$ (T.V), followed by its inverse function $p(k, \alpha)$ tabulated for $k = 0(.05)1$ and $\alpha = 0(.01)1$ to 4D (T.VI). The main and most valuable feature of this number is, however, a group of 4D-tables of $p(k, \alpha'^{x})$ (T.VII–X) and $p(k, \alpha''^{x})$ (T.XI–XIV) relevant respectively to occultation and transit eclipses and each evaluated for $x = .2(.2).8$, for the same range and subdivision of the arguments as T.VI.

The concluding part of No. 50 is then devoted to the tables pertaining to annular eclipses. If the eclipse is annular, Ţsesevich replaces, for reasons of convenience, the geometrical depth p of the eclipse by $q = [k(1 + p)]/(k - 1)$, defines

$$\alpha''^{x}(k, q) = 1 + A^{x}(k)X(k, q),$$

and tabulates $X(k, q)$, to 5D, as a function of $k = .2(.05)1$ and $q = 0(.01)1$ (T.XV) as well as its invese $q(k, X)$, likewise to 5D, as a function of k and X for the same range and subdivision of the argument (T.XVI). T.XVIa, an expanded portion of T.XVI, gives the tabular values of $q(X, k)$ for $k = .2(.1)1$ and $X = .85(.005).95(.001).995(.0005).998(.0002).999-(.0001)1$. No. 50 concludes with five univariate tables of $A^{x}(k)$ evaluated to 5D for $k = .2(.01)1$ and $x = 1(-.2).2$ (T.XVII–XXI). T.XVII giving $A(k)$ contains also first and second differences.

Ţsesevich's work leaves at present little to be desired and should probably be regarded as the standard set of fundamental tables that are likely to meet all astronomical requirements for a long time to come. The author did not give the reader as much insight into the processes by which his tables were constructed as one might wish to have, but they appear to have been essentially sound and their results impressive. Skeleton tables of all α-functions were apparently first computed to 6D; the tables enumerated above were obtained from them subsequently by interpolation and rounding off. Numerous and careful spot-checks of Ţsesevich's tables by Professor JOHN E. MERRILL of Hunter College, and by the present reviewer, disclosed that in general Ţsesevich's tables can be trusted to the number of digits given in his tables; rarely do departures from true values amount to more than one unit of the last place. In *Bulletin of the Panel on Orbits of Eclipsing Variables*, no. 2 (Harvard College Observatory, June 1946, p. 8–11), Merrill lists 42 errors exceeding 2 units of the last place found in Ţsesevich's tables, and his list should be consulted by anyone planning to make extensive use of them. Misprints in Ţsesevich's tables are, unfortunately, many; most of them can, however, be easily detected from the behavior of local tabular differences. A systematic survey of tabular differences of Ţsesevich's tables would undoubtedly disclose a great many more misprints than have so far been noticed, and is much to be desired. Apart from misprints, however, the printing of the tables is generally good and clear, though the paper, especially that of no. 45, is of poor quality. Both issues are provided with brief English summaries of their contents and also all headings are in Russian and English; but the wealth of information regarding the mathematical character of the tabulated functions, contained in the introduction to each of the issues, is accessible only to those who read Russian. It may be mentioned that no. 50 was published in Leningrad shortly before the outbreak of Russo-German war and apparently few copies were distributed before Leningrad became a theatre of war. The reviewer learns, however, through Professor SHAPLEY, that the stock of this valuable publication survived intact and is now available for distribution.

A comparison of Ţsesevich's 5S tables with the 4S tables of $\alpha'(k, p)$ and $\alpha''(k, p)$, published earlier by Ferrari,[8] affords a convincing check of the accuracy of the later numerical work. It should be pointed out that Ferrari did not evaluate the integrals expressing the α's in terms of the elliptic functions, but computed them with the aid of GAUSS's formula for approximate quadratures. Ferrari himself found that his series converged rather slowly, and the correction factor he obtained, by a comparison of the few entries directly computed for the transit, did not bring the systematic errors within satisfactory limits in many regions.

Entry-by-entry check of the 4S Ferrari against 5S Tsesevich values rounded off to 4D, performed by Merrill (*l.c.*), disclosed that about three-fourths of Ferrari's values of $\alpha'(k, p)$ and one-half of his $\alpha''(k, p)$'s are a unit or more in error. Ferrari's α''s are systematically too high (sometimes as much as 3 units) for positive p's and too low (frequently by 3 units) for negative p's. His α'''s are, in general, about one unit too low for large values of k and large positive p's one unit too high for k about 0.55 and large positive p's, as much as 4 units too low around $k = 0.55$ and $p = -0.85$, and too high by as much as 6 units around $k = 0.6$ and $p = -0.65$. Inasmuch as the appearance of more extensive and accurate tables by Tsesevich has rendered all of Ferrari's work rather out of date, it does not seem worth while to publish any detailed correction table to his results.

The mathematical definitions of the tables discussed above may now be set forth. The loss of light f arising from the eclipse of an arbitrarily darkened disc obeys the equation (1) above, where

$$\pi r_1^2 f^0 = \left\{\int_s^{r_1}\int_{-(r_1^2-x^2)^{\frac{1}{2}}}^{(r_1^2-x^2)^{\frac{1}{2}}} + \int_{\delta-r_2}^{s}\int_{-[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}\right\} dxdy \tag{2}$$
$$= r_1^2\cos^{-1}s/r_1 + r_2^2\cos^{-1}(\delta-s)/r_2 - \delta(r_1^2-s^2)^{\frac{1}{2}}$$

if the eclipse is partial, and

$$\pi r_1^2 f^0 = \int_{\delta-r_2}^{\delta+r_2}\int_{-[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2^2-(\delta-x)^2]^{\frac{1}{2}}} dxdy = \pi r_2^2 \tag{3}$$

if it is annular; whereas

$$\pi r_1^3 f^D = \frac{3}{2}\left\{\int_s^{r_1}\int_{-(r_1^2-x^2)^{\frac{1}{2}}}^{(r_1^2-x^2)^{\frac{1}{2}}} + \int_{\delta-r_2}^{s}\int_{-[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}\right\}(r_1^2-x^2-y^2)^{\frac{1}{2}}dxdy, \tag{4}$$
$$= \pi r_1^3 - 2r_1^3\{(E-F)F' + E'F\} + [r_2(\delta/r_2)^{\frac{1}{2}}/3]\{7r_2^2 - 4r_1^2 + \delta^2\}(2E-F)$$
$$- [(\delta/r_2)^{\frac{1}{2}}/3\delta]\{5\delta^2r_2^2 + 3(r_1^2-r_2^2)^2 - 3\delta r_1^2\}F$$

if the eclipse is partial, and

$$\pi r_1^3 f^D = \frac{3}{2}\int_{\delta-r_2}^{\delta+r_2}\int_{-[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}(r_1^2-x^2-y^2)^{\frac{1}{2}}dxdy, \tag{5}$$
$$= 2r_1^3\{(E-F)F' + E'F\} + \tfrac{1}{3}\{7r_2^2 - 4r_1^2 + \delta^2\}[2\delta(r_2-\delta+s)]^{\frac{1}{2}}E$$
$$+ \tfrac{1}{3}\{(\delta^2-r_2^2)^2 - r_1^2(2r_1^2-r_2^2) + 6\delta r_1^2\}F/[2\delta(r_2-\delta+s)]^{\frac{1}{2}}$$

if it is annular. In these latter equations, $F = F\left(\frac{\pi}{2}, k\right)$ and $E = E\left(\frac{\pi}{2}, k\right)$ denote the LEGENDRE complete integrals of the first and second kinds, with the modulus $k^2 = \frac{1}{4}[1 - (\delta - s)/r_2]$ for a partial eclipse and $= 2[1 - (\delta - s)/r_2]^{-1}$ for an annular eclipse. $F' = F(\phi, k')$ and $E' = E(\phi, k')$ stand for the incomplete integrals of the same kind with a complementary modulus $k' = (1-k^2)^{\frac{1}{2}}$ and an amplitude $\phi = \sin^{-1}[2\delta/(r_1+r_2+\delta)]^{\frac{1}{2}}$ if the eclipse is partial, and $\phi = \sin^{-1}[(r_1+r_2-\delta)/(r_1+r_2+\delta)]^{\frac{1}{2}}$ if it is annular.

For the purposes of tabulation, we prefer to normalize the f-functions so as to make them vary between 0 at the moment of the beginning of the eclipse ($\delta = r_1 + r_2 = r_a + r_b$) and 1 at the moment of internal tangency of both discs ($\delta = r_a - r_b$). This can be achieved by putting, by definition, $f^0 = \alpha(k, p)$, if $r_1 = r_a$; or $f^0 = k^2\alpha(k, p)$ if $r_1 = r_b$. $f^D = \alpha'(k, \cdot)$ if $r_1 = r_a$; or $f^D = \frac{3}{4}k^2\Phi(k)\alpha''(k, p)$, if $r_1 = r_b$, where

$$\Phi(k) = 4\{\sin^{-1}\sqrt{k} + \tfrac{1}{4}(4k-3)(2k+1)\sqrt{k(1-k)}\}/3\pi k^2. \tag{6}$$

$\Phi(k)$ is equal to $\frac{4}{3}k^2$ times the value of f^D at the moment when $\delta = r_1 - r_2$. Tsesevich tabulates $\Phi(k)$ to 6D for $k = .2(.01)1$ (no. 50, p. 307, auxiliary T.5). The function $\frac{4}{3}k^2\Phi(k)$ was also previously tabulated by Ferrari [6] to 4D for $k = .2(.05)1$. (v. 148, p. 221).

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[1] For their details see, for instance, Z. KOPAL, *Introduction to the Study of Eclipsing Variables* (*Harvard Observatory Monographs*, no. 6), Cambridge, Mass., 1946, chapter 2.

[2] This table was later recomputed by B. W. SITTERLY, Princeton Univ. Observatory, *Contributions*, no. 11, 1930, 41 p. 23 × 32 cm.

[3] According to TSESEVICH, no. 45, p. 124, "T.Iy [p. 390] is especially deficient in accuracy and contains systematic departures from accurate figures by twenty and more units of the third decimal."

[4] A. G. M. WEND, *Eine Tafel zur Theorie der Bedeckungsveränderlichen*. Diss. Leipzig. Weida i. Thür, 1931, 44 p. 17.7 × 26.3 cm.

[5] E. M. R. HETZER, *Beitrag zu H. N. Russell's Methode der Berechnung der Elemente von Verfinsterungsvariablen unter Voraussetzung von Kreisbahnen und gleichmässig hellen Sternscheiben*. Diss. Leipzig. Weida i. Thür, 1931, 56 p. 23.7 × 29 cm.

[6] K. G. FERRARI, "Zur Theorie des Bedeckungslichtwechsels bei vollständig randverdunkelten Sternscheiben." Akad. d. Wissen., Vienna, *Abt.* IIa, (i) v. 147, 1938, p. 497–511; (ii) v. 148, 1939, p. 217–235.

At the time of publication of Ferrari's memoirs, only a few skeletons of Tsesevich's tables of $\alpha'(k, p)$ and $\alpha''(k, p)$ were available for comparison; see Pulkovo, Astron. Observatoriia, Tsirkuliary [Circulars], no. 24, 1938, p. 41–45.

549[U].—JAPAN, HYDROGRAPHIC OFFICE, *Publication* no. 601. *Celestial Navigation Computation Tables*. Tokyo, Hydrographic Office, 1942. 138, xxix p. 18 × 25.3 cm.

These tables were intended for surface navigation; the volume contains practically everything that will be needed for celestial navigation beyond the material in a Nautical Almanac.

The reader will find the title page of this volume facing page 138! It is followed by a table of contents and twenty-seven pages of explanation including one page giving English equivalents of a number of Japanese characters appearing frequently in the text.

The first fourteen pages of the volume are given over to the usual auxiliary tables of navigation, conversion of time to arc, and corrections to be applied to observed altitudes. The latter are unusual in that they include corrections to be applied to altitudes less than 6°; special corrections for temperature, barometric pressure, and difference between air and water temperatures are given.

The basic table provided for the computation of the altitude and azimuth of a celestial body (no. 6, p. 15–79) is similar to that of YONEMURA [*Tables for calculating Altitudes and Azimuths of Celestial Bodies*, Tokyo, 1920]. In the usual American notation (used throughout this review) where t, d are the LHA and declination of the celestial body, L the latitude of the observer, and h, Z the altitude and azimuth angle of the celestial body, the formulae are (θ being an auxiliary angle)

$$\log (1/\text{hav } \theta) = \log (1/\text{hav } t) + \log \sec d + \log \sec L,$$

$$\text{hav } (90° - h) = \text{hav } (L \pm d) + \text{hav } d, \log \csc Z = \log \csc t + \log \sec d - \log \sec h.$$

The table gives the values, to the nearest integer, of $10^5 \log (1/\text{hav } x)$ and $10^5 \log \csc x$ for $x = 0(1')360°$; and of $10^5 \text{ hav } (90° - x)$, $10^5 \text{ hav } x$, and $10^5 \log \sec x$ for $x = 0(1')90°$.

Table 7 gives the altitude to the nearest 0°.1 of a celestial body of declination $d = 0(1°)10°(2°)62°$ when it is on the prime-vertical at latitude $L = 0(1°)50°(2°)70°$.

Table 8 is designed for the computation of azimuth angle by means of the formulae:

$$\tan \tfrac{1}{2}(Z + q) = \cos \tfrac{1}{2}(L + d) \csc \tfrac{1}{2}|L - d| \tan \tfrac{1}{2}t,$$
$$\tan \tfrac{1}{2}(Z - q) = \sin \tfrac{1}{2}(L + d) \sec \tfrac{1}{2}(L - d) \tan \tfrac{1}{2}t.$$

with argument $S = L + d = 0(5')6°(10')90°$, the values of $X_1 = 3200 + 10^3 \log \cos \tfrac{1}{2}S$ and $Y_1 = 3200 + 10^3 \log \sin \tfrac{1}{2}S$ are given to the nearest integer. With argument $D = |L - d| = 0(5')6°(10')174°(5')180°$, the values of $X_2 = 10^3 \log \csc \tfrac{1}{2}D$; $Y_2 = 10^3 \log \sec \tfrac{1}{2}D$ are given to the nearest integer. With argument $t = 0(5')6°(10')174°(5')188°(10')354°(5')360°$, the values of $X_3 = Y_3 = 3200 + 10^3 \log \tan \tfrac{1}{2}t$ are given to the nearest integer. With argument x (or y) $= 0(0°.05)3°(0°.1)87°(0°.05)90°$, the values of $X_4 = 6400 + 10^3 \log \tan x$ (or $Y_4 = 6400 + 10^3 \log \tan y$) are given to the nearest integer. This table is especially con-

venient for the calculation of a series of azimuths of a celestial body for a fixed position of the observer and constant declination of the celestial body.

Table 9 is intended for star identification, that is, one can start with values of h, Z and L, and quickly obtain values of d and t, usually, enough to identify the celestial object. The formulae used are:

$$\cot K = \sec Z \tan h, \sin d = \sin h \sec K \sin (L + K), \tan t = \tan Z \sin K \sec (L + K).$$

With arguments $Z = 0(1°)180°$, $h = 0(1°)90°$, $K = 0(1°)90°$, $L + K = 0(1°)180°$, $d = 0(1°)90°$, $t = 0(1°)90°$, the values of $K_1 = 10^3 \log \sec Z$, $H_1 = 3000 + 10^3 \log \tan Z$, $K_2 = 3000 + 10^3 \log \tan h$, $D_1 = 3000 + 10^3 \log \sin h$, $K_3 = 3000 + 10^3 \log \cot K$, $D_2 = 10^3 \log \sec K$, $H_2 = 3000 + 10^3 \log \sin K$, $D_3 = 3000 + 10^3 \log \sin (L + K)$, $H_3 = 10^3 \log \sec (L + K)$, $D_4 = 6000 + 10^3 \log \sin d$, $H_4 = 6000 + 10^3 \log \tan t$ are given to the nearest integer. This description makes the tables sound cumbersome; actually they occupy only 3 pages and the formulae are written $K_1 + K_2 = K_3$; $D_1 + D_2 + D_3 = D_4$; $H_1 + H_2 + H_3 = H_4$.

Table 11A is essentially a "distance-travelled" table for use with hourly rate 6(.1)34 knots for time intervals ranging from 1^m to 10^d, with tabulated value in nautical miles. Table 11B yields distance travelled in meters in 1 to 60 seconds at speed ranging from 1 to 34 knots. Table 12 is a table of Difference of Longitude and Departure for courses 1°(1°)89° corresponding to distances of 0(1′)100′(100′)900′. Table 13 gives difference of longitude to 0.1 with arguments departure of 0(1′)10′(10′)100′(100′)900′ and middle latitude 0°, 4°(2°)-10°(1°)65°. Table 14 is one of meridional parts for latitudes 0(10′)90° with proportional parts for interpolation; it is based on Bessel's figures and hence now out of date. Tables 15–22 are respectively 4-place tables of $\log N$, $\log \sin x$ for $x = 0(1')10°$; $\log \tan x$ for $x = 10'(1')10°$; $\log \sin x$ for $x = 0(10')90°$; $\log \tan x$ for $x = 10'(10')89°50'$; $\sin x$ for $x = 0(10')90°$; $\tan x$ for $x = 0(10')89°50'$; $\sec x$ for $x = 0(10')89°50'$. Tables 23–25 include mathematical formulae and numerical values likely to be of use to mariners. Table 26 is one of equivalents —weights, measures, areas, etc.

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550[U].—JAPAN, HYDROGRAPHIC OFFICE, *Publication* no. 602. *Brief Celestial Navigation Table* (*Dead Reckoning Position Method*). Tokyo, Hydrographic Office, October 1942. 4, 50, xxiv p. 18.2 × 25.7 cm.

This volume presents a dead-reckoning position method based on a division of the astronomical triangle into two right triangles by a perpendicular dropped from the celestial body upon the meridian. K is defined in this book as the polar distance of the foot of the perpendicular. Using this, and the usual American notation (as indicated in the preceding review), the equations are:

$$\log \tan K = \log \cot d + \log \cos t$$
$$\log \cot Z = \log \cot t + \log \csc K + \log \cos (K + L)$$
$$\log \cot h = \log \cot (K + L) + \log \sec Z.$$

These are written $K_3 = K_1 + K_2$, $Z_4 = Z_1 + Z_2 + Z_3$, $A_3 = A_1 + A_2$. This notation makes the computing form simple, but it involves a certain amount of duplication in the tables provided. There are three tables, each with values given to four decimals. In the first, $\log \cot x$ is given for $x = 0(1')90°$. The second table gives $\log \cot x$ and $\log \cos x$ for $x = 0(1')360°$. The third table gives $\log \tan x$, $\log \csc x$, $\log \cot x$, $\log \sec x$ for $x = 0(1')90°$. These tables remind one of the four holes the hired man cut in the barn door for the mother cat and her three kittens.

The title page follows page 50 and is in turn followed by twenty pages of explanation in Japanese. The usual auxiliary tables giving corrections to observed altitudes are given in graphical form on the first four pages of the volume; they are also given in the usual tabular form (but to a low order of accuracy) on page 50. A multiplication and interpolation table

for hourly differences 2(1)20 allows the determination of values corresponding to time intervals ranging from one second to eight days. Following this, there is a table giving the north–south and east–west components of distances 1′ to 200′ along courses 0 to 90°. Another table gives the longitude differences corresponding to distances (measured in an east–west direction) 1′(1′)14′ at latitudes 0°, 4°, 8°, 10°, 12°, 14°(1°)70°.

The design of this volume would indicate it was planned for men with a very limited background in celestial navigation. For this reason, it might have a strong appeal for many of the persons who now use Martelli's tables. A simple trimming of the top and bottom corners of pages helps one to locate the desired tables quickly.

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551[V].—MASSACHUSETTS INSTITUTE OF TECHNOLOGY, Department of Electrical Engineering, Center of Analysis, Technical Report no. 3, work performed under the direction of ZDENĚK KOPAL, under NOrd Contract No. 9169: *Tables of Supersonic Flow Around Yawing Cones.* Cambridge, Mass., 1947, xviii, 324 p. +2 folding plates, 20.2 × 26.8 cm. For sale by Superintendent of Documents, Washington, D. C., $2.50.

The supersonic flow of air around a non-yawing cone has been numerically solved and presented in rather complete tables entitled *Tables of Supersonic Flow around Cones* (RMT 475, *MTAC*, v. 3, p. 37–40). Such information is very valuable in considering the performance of sharp-nosed projectiles or rockets when flying directly along the axis. Any device, however, which moves in free flight through the atmosphere can be maintained only approximately in a non-yawed position. Since yawing motion is inevitable, additional questions are raised which cannot be answered from the previously published tables. The first question of importance is the sensitivity of the drag to yaw; that is, will the forces on the nose of the projectile change appreciably with small angles of yaw? The second and somewhat more important question which is raised is that concerned with the normal force. Just as an airplane wing flying at an angle of attack will experience a lift force normal to its direction of motion, so a cone flying yawed at supersonic speed will experience a force normal to its direction of motion. Such normal forces, if large, would make the problem of control of the flight of a body difficult, since these forces must be properly counterbalanced if the motion is to be stable.

In the present work, the first-order improvement to the former tables is given. The angle of yaw ϵ is considered so small that its square and all higher powers can be ignored. With this approximation, the general equations of motion in polar coordinates are reduced to the following non-homogeneous, second-order, ordinary differential equation with non-constant coefficients:

$$\frac{d^2(x/d_1)}{d\theta^2} + A\frac{d(x/d_1)}{d\theta} + B(x/d_1) + C = 0.$$

In this equation x gives the correction to the radial velocity as a function of the polar angle θ. The coefficients A, B, C are rather intricate functions of θ, as well as of various quantities occurring in the solution for a non-yawed cone. Specifically, the two-velocity components, the speed of sound, the pressure, density, and ratio of specific heats all enter. As is noted in the text, these expressions are obtained after "requisite and rather troublesome algebra." The constant d_1 in this equation is evaluated from the boundary conditions at the shock wave.

In this work, while approximations are made appropriate to a slight yawing of the cone, there are no approximations made relative to the shock wave. The flow behind the shock wave is treated as rotational and of non-constant entropy as is appropriate to this problem. In this respect the present work, which follows the theory of STONE,[1] is an improvement on the earlier work of TSIEN [2] and SAUER.[3]

To obtain a solution an initial Mach number and shock wave angle θ_w were chosen, and the integration was carried out numerically toward decreasing θ until the appropriate boundary condition at the surface of the cone was obtained. From the resulting values of x all other information could then be simply computed. In particular, the yaw of the shock wave which is in general not the same as that of the cone, was found. In all of this work it was assumed that the Mach number of the free stream was high enough for the shock wave to be attached to the vertex of the cone. The results of the computation are given in sufficient detail to compute the variation in the three-velocity components and the pressure and density around a slightly yawing cone. In addition, normal force coefficients and drag coefficients are given. It is noted, however, that to the present approximation the drag coefficients are identical with those for a non-yawed cone as reported previously.

In three tables, included in the introduction, $\theta_s = 10°(5°)20°$, $\bar{u}_s = .4(.1).9$, a comparison is made of calculations of the normal force coefficients by use of Tsien's formula and the present tables. The error ranges from -12.2% to $+44.2\%$ (ignoring the fact that Tsien's formula for sufficiently high Mach number gives imaginary results). The tables presented are as follows:

I, p. 1–306: *Tables of Supersonic Flow of Air Around Slightly Yawing Cones.*

These tables give, successively, the contributions x, y, z to the velocity components u, v, w taken in the direction of increasing spherical polar coordinates r, θ, ϕ, due to a small yaw, as functions of the angular variable θ; and the corresponding proportional changes in pressure $(\eta/\bar{p})$ and density $(\xi/\bar{\rho})$ due to the same cause.

The velocities are expressed in terms of the velocity which the air in front of the shock wave would attain if allowed to expand adiabatically into a vacuum.

These 4D tables are for $\theta_s = 5°(2°.5)15°(5°)50°$, $\theta = 5°$ to $89°.612$ in steps ranging from $.125°$ to $1°$. These cover the entire range of interesting cone angles and speeds including some cases of the "second" solution, i.e., strong shock (see previous review).

II, p. 307–312: *Survey of the Results.* The individual columns indicate:

$\bar{u}_s$ = radial velocity-component of the axial flow along the solid surface;
θ_w = semi-apex angle of the shock wave;
M = Mach number (i.e., stream velocity divided by the velocity of sound in the undisturbed air in front of the shock wave);
d_1 = constant specifying the change of entropy across streamlines;
δ/ϵ = ratio of the yaw of the shock wave to the yaw of the solid cone;
K_D = coefficient of head drag on the cone;
K_N = coefficient of normal drag on the cone.

III, p. 315–317: *Table Giving the Shock-to-Cone Yaw Ratio,* δ/ϵ, in Terms of the Radial Velocity Component, u_s, along the Solid Surface, and the Semi-Apex Angle of the Cone, θ_s.

The values in this table are primarily 4D results interpolated for more convenient use from the previous computations.

IV, p. 319–321: *Table Giving the Coefficient of Normal Drag,* K_N, in Terms of the Radial Velocity Component, $\bar{u}_s$, along the Solid Surface, and the Semi-Apex Angle of the Cone θ_s.

Here again 4D interpolated values are given for convenience.

V: *Two folding charts* are given which show the yaw of the shock wave and the normal force coefficients as functions of the cone angle and velocity along the cone. These give a good general picture of the nature of the flow and make for ready practical use where high accuracy is not required.

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[1] Arthur H. Stone, *The Aerodynamics of a Slightly Yawing Supersonic Cone,* NDRC Report, Div. 1, 1944.

[2] Hsue-Shen Tsien, "Supersonic flow over an inclined body of revolution," *Jn. Aero. Sci.*, v. 5, 1938, p. 480.

[3] R. Sauer, "Überschallströmung um beliebig geformte Geschosszspitzen unter kleinem Anstellwinkel," *Z. f. Luftfahrtforschung,* v. 19, 1942, p. 148–152.

552[V].—National Advisory Committee for Aeronautics, *Technical Note* no. 1428: *Notes and Tables for Use in the Analysis of Supersonic Flow*. By the staffs of the Ames Aeronautical Laboratory, Moffett Field, Calif. Washington, D. C., Dec. 1947, iv, 73 p. +7 plates. 20 × 26.2 cm.

This NACA *Technical Note* is a compilation of data found important for the analysis of compressible flow in connection with test work in a supersonic wind-tunnel. The text of the paper is devoted to a review of several fundamental aspects of the theory of supersonic flow. These include thermodynamics, equations of motion, nozzle theory, shock waves, expansion around a corner, airfoil theory, and flow about wedges and cones. The appendices give formulae for the calculation of the viscosity, Reynolds number, humidity relations, and atmospheric corrections for air.

The five tables included in the paper contain the following information: **Table I** gives various nozzle data for subsonic flow such as the ratios of the local pressure to rest pressure, the local density to the rest density, etc. for values of the Mach number. The Mach number $M = 0(.01)1$. **Table II** gives similar nozzle data for supersonic flow and in addition gives such data as the local Mach angle, the angle through which a supersonic stream is turned to expand from Mach number 1 to the local Mach number, etc. $M = 1(.01)3.5(.1)5(1)10(5)20, 100, \infty$. **Table III** gives ratios of the local pressure to the rest pressure, the local density to the rest density, etc. on either side of a normal shock wave. **Table IV** gives the Mach number functions for use with small perturbation airfoil section theory, and **Table V** gives various properties of the standard atmosphere. All of the data given in the tables pertain to air where the ratio of the specific heats is 1.4. The functions are tabulated to at least 4S accuracy.

The graphs that are given include plots of maximum theoretical contraction ratio that permits start of supersonic flow in diffuser entrance against Mach number, and the variation of Reynolds number with Mach number. The well-known graphs for the characteristics of wedge and cone flow are also included.

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MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **521** (Lambert), **522** (Peters), **526** (Rybner), **529** (Kraïtchik), **531** (Pettit), **535** (Cambi), **541** (Lane & Sweeney), **548** (Ferrari, Russell & Shapley, Tsesevich).

128. J. R. Airey, "Tables of the Bessel functions $J_n(x)$," BAAS, *Report*, 1915, p. 29–30.

A list of errors was given in MTE 124 for the 10-decimal part of this table for $n = 0(1)13$ where $x > 6$. The whole table has now been compared with proofs of the forthcoming BAAS *Bessel Functions*, part 2, and the following 23 further errors have been found for [$x = .2(.2)6$; 6D]. Thus the total number of errors, large and small, in this table is 74.

x	n	*For*	*Read*	x	n	*For*	*Read*
1.8	9	—	+0.000 001	5.6	9	+0.012 893	+0.012 907
2.4	10	+0.000 002	+0.000 001		10	+0.003 870	+0.003 912
2.6	9	+0.000 024	+0.000 025		11	+0.000 930	+0.001 062
2.8	11	—	+0.000 001		13	+0.000 057	+0.000 059
3.8	11	+0.000 021	+0.000 022	5.8	4	+0.378 765	+0.378 766
	12	+0.000 004	+0.000 003		8	+0.046 382	+0.046 381
4.6	13	+0.000 005	+0.000 006		9	+0.016 641	+0.016 639
5.4	5	+0.310 074	+0.310 070		10	+0.005 261	+0.005 256
	13	+0.000 037	+0.000 038		11	+0.001 500	+0.001 486
5.6	6	+0.198 559	+0.198 560		12	+0.000 380	+0.000 381
	7	+0.094 452	+0.094 455		13	+0.000 088	+0.000 089
	8	+0.037 571	+0.037 577				

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6 Feb. 1948

129. G. F. BECKER & C. E. VAN ORSTRAND, *Hyperbolic Functions* (*Smithsonian Mathematical Tables*), Fifth reprint, 1942. See *MTAC*, v. 2, p. 311.

Comparison of this volume with recent publications [1] and the original sources of the tables [2] revealed the discrepancies listed in A. Errors and misprints; B. Errors in excess of 5 units in the next succeeding place of decimals or significant figure.

A.	Page	u	Function	*For*	*Read*
	xvi		equation 86	+	−
	18	0.0803	log tanh u	.90380	.90378
	18	0.0803	log coth u	.09620	.09622
	29	0.485	$\omega F_0'$	65.5	96.5
	40	1.009	log cosh u	.18137	.19137
	46	1.339	log tanh u	.04023	.94023
	60	2.018	log sinh u	.56723	.56763
	125	0.980	$\omega F_0'$	144	114
	155	2.474	$\omega F_0'$	498	598
	182	0.0414	u	.0411	.0414
	212	1.027	u	3.3.96	33.96
	214	1.117	u	5.7.79	57.79
	238	1.217	e^{-u}	.296 1772	.296 1172
	245	1.976	u	.970	.976
	253	2.709	e^{-u}	.066 6039	.066 6034
	261	0.096	p	2169	4169
	270		u	2959	2939
	270		u	3777	3877
	271		u	5421	5431
	272	7853	ln u	8.96765	8.96865
	288	1.242	gd u	.008 4840	.008 4849
	293	1.730	gd u	37.90	37.00
	306	3.59	$\omega F_0'$	113.66	113.76
	307	4.32	gd u	80 28 33.73	88 28 33.73
	310	23	$\mathrm{gd}^{-1} u$	505.53	565.53

B.	Page	u	Function	*For*	*Read*
	32	0.631	log cosh u	.08126	.08127
	34	0.703	log tanh u	.78266	.78267
	34	0.703	log coth u	.21734	.21733
	91	0.0157	coth u	63.699	63.700
	91	0.0166	coth u	60.247	60.246
	100	0.0635	tanh u	.06342	.06341
	107	0.0972	tanh u	.09689	.09690
	107	0.0982	tanh u	.09788	.09789
	109	0.164	tanh u	.16254	.16255
	111	0.254	tanh u	.24867	.24868
	123	0.876	sinh u	.99241	.99242
	130	1.242	tanh u	.84602	.84603
	131	1.263	sinh u	.62661	.62660
	132	1.319	tanh u	.86653	.86654
	133	1.399	tanh u	.88513	.88514
	134	1.449	tanh u	.89550	.89549
	138	1.624	cosh u	.63522	.63523
	141	1.775	tanh u	.94416	.94415
	141	1.784	tanh u	.94513	.94512
	143	1.876	tanh u	.95414	.95413
	143	1.891	sinh u	.23753	.23754
	144	1.932	tanh u	.95890	.95889
	145	1.972	tanh u	.96199	.96200
	171	5.60	cosh u	135.2150	135.2151
	186	0.0627	cos u	.99804	.99803
	212	1.015	$\omega F_0'$	85.0	84.9
	216	1.229	u	59″.44	59″.45
	224	2.6	u	08″.49624 24	08″.49624 25
	224a	7.8	u	25″.48872 73	25″.48872 74
	237	1.187	log e^u	.515 5075	.515 5076
	251	2.521	e^u	12.441 032	12.441 031
	252	2.626	log e^u	1.140 4572	1.140 4573
	259	42	e^u	173 927 493	173 927.494

The values of tanh 0.174, cosh 0.911, tanh 0.932, tanh 1.381, tanh 1.986 cannot be rounded off correctly with existing tables. Following are three exceptional cases:

$$\sinh 0.876 = 0.99241\ 5002; \qquad \cos 0.0627 = 0.99803\ 4999;$$
$$\sin 0.394 = 0.38388\ 49999\ 93636.$$

The error of 5 units in the seventh decimal of $e^{-2.709}$ had its origin in a misprint in Newman's table; see *MTAC*, v. 1, p. 455.

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[1] F. E. PERNOT & B. M. WOODS, *Logarithms of Hyperbolic Functions to twelve Significant Figures*. Berkeley, Cal. 1918. (Univ. California, Publs. in Engineering, v. 1, no. 13.)
NBSCL, (a) *Tables of the Exponential Function* e^x. 1939;
(b) *Tables of Sines and Cosines for Radian Arguments*. 1940;
(c) *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*. 1940;
(d) *Tables of Natural Logarithms*, v. 1, 1941;
(e) *Tables of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*. 1943.
[2] C. GUDERMANN, *Theorie der Potenzial- oder cyklisch-hyperbolischen Functionen*. Berlin, 1833.
J. INMAN, *Nautical Tables designed for the Use of British Seamen*. Rev. by J. W. INMAN, London, 1867, p. 364–372.
F. W. NEWMAN, "Table of the descending exponential function to twelve or fourteen places of decimals," Cambridge Phil. Soc., *Trans.*, v. 13, part 3, 1883, p. 145–241.
J. W. L. GLAISHER, "Tables of the exponential function," Cambridge Phil. Soc., *Trans.*, v. 13, part 3, 1883, p. 243–272.
J. O. W. LIGOWSKI, *Tafeln der Hyperbelfunktionen und der Kreisfunktionen*. Berlin, 1889.
A. FORTI, *Nuove Tavole delle Funzioni Iperboliche*. Rome, 1892.

130. H. B. DWIGHT, *Mathematical Tables . . .*, 1944. See *MTAC*, v. 1, p. 180.

P. 45, sec 29°.62 to 29°.67 inclusive, *change* the third decimal figure, 1, in each of the six cases, *to* 0.

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131. H. B. DWIGHT, *Tables of Integrals and Other Mathematical Data*, Revised edition. 1947. See *MTAC*, v. 2, p. 346.

P. 44, formula 195.01, first line, *for* $bg < 0$, *read* $b > 0$, $g < 0$.

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132. C. F. GAUSS, *Werke*, v. 3, 1866, zweiter Abdruck, 1876, "De curva lemniscata," p. 413–432. This item is in "Nachlass," first published in this v., 1866.

This second posthumous paper by Gauss on the subject of lemniscate functions is concerned principally with the expansion of these functions in trigonometric series.

Certain of the basic constants appearing in the coefficients of these series were evaluated by Gauss to as many as 57D. Recently I recalculated to a higher degree of approximation all the numerical data in this paper. A number of serious calculational errors, in addition to several minor terminal-digit inaccuracies, were thus discovered in the work of Gauss. Among these is a fundamental error in Gauss's approximation to the quarter-period $\tilde{\omega}/2$ of the lemniscate functions.

For convenient reference, the errors and their corrections are presented in the following tabular array.

Page	Constant	*For*	*Read*
413	$\tilde{\omega}/2$	680 320(7)	523 2420
	$(\tilde{\omega}/2)^3$	11(7)	207
	$(\tilde{\omega}/2)^4$	63(4)	942
	$(\tilde{\omega}/2)^5$	70 0712625(4)	66 04823170
	$(\tilde{\omega}/2)^6$	025(3)	4275
	$(\tilde{\omega}/2)^8$	569(0)	7567
	$(\tilde{\omega}/2)^9$	59	68
	$(\tilde{\omega}/2)^{12}$	1749	8915
	$(\tilde{\omega}/2)^{12}$	4025	7405
414	$\log(\tilde{\omega}/2)$	26 9692(2)	45 79920
	$\ln(\tilde{\omega}/2)$	0 6425	2 6427
	$\ln\tilde{\omega}$	2 36577	4 36590
418	$\frac{\pi}{12} - \frac{1}{2}\ln 2$	372 7	024 8
	$\operatorname{csch}\pi$	1 57	5 30
	$\frac{1}{2}\operatorname{csch} 2\pi$	144 52	492 44
	$\frac{1}{2}\operatorname{csch} 3\pi$	86	79
	$\frac{1}{2}\operatorname{csch} 7\pi$	81	80
	$2/(e^{2\pi} - 1)$	98	97
	$1/(e^{4\pi} - 1)$	54	55
	$2e^{-9\pi/4}$	906	907
	$2e^{-25\pi/4}$	31 18	45 82
419	$2e^{-\pi}$	514	515
	$2e^{-4\pi}$	387 96535	532 53042
	$2e^{-9\pi}$	288	290
420	$A = \sqrt{\frac{\pi}{\tilde{\omega}}}\Big/(2^{7/4}\cos\pi/8)$	6226 6118372314	8226 6118369169
	$2Be^{-\pi/2} = 2e^{-\pi/2}(A\cot\pi/8)$	19576 3585935635	21576 3585929459
	$2Ae^{-2\pi}$	9042	5000
	$2Be^{-9\pi/2}$	6398	5261
	$2Ae^{-3\pi}$	170	051
	$2Be^{-25\pi/2}$	94	99
421	$340-152\sqrt{5}$	01228	01221
	$\sqrt{340-152\sqrt{5}}$	7310507555 85731	6847092228 86965
	$(\sin\operatorname{lemn}\frac{1}{4}\tilde{\omega})^4$	7731504831 80010	8194920158 78776
	$(\sin\operatorname{lemn}\frac{1}{4}\tilde{\omega})^4$	2352519943 51472	1889104616 52706
	$(\sin\operatorname{lemn}\frac{1}{4}\tilde{\omega})^2$	710	711
	$(\sin\operatorname{lemn}\frac{1}{4}\tilde{\omega})^2$	029	052
423	$N = 2(\tilde{\omega}/\pi)^2$	203	204
	$a = (2/\tilde{\omega})^2$	803	802
	$(\ln 2)/\pi$	635	636
428	$e^{-\pi}$	636	633
429	$e^{-\pi/4}$	366	365
430	$e^{-\pi/4}$	820	785
431	$e^{-9\pi/4}$	4846487994 1872486024 8176915	6000857699 5032475616 1924591
431	$e^{\pi/2}$	44648993 1536	35666703 8331

The third calculation of $e^{-\pi/4}$, shown on p. 430, lacks one step for completion. The work to that point is correct to 48D.

Because of an error affecting the last four decimal places in the second line of work, the second calculation of $e^{\pi/2}$, on p. 432, when completed yields a result correct to only 30D.

An incorrect algebraic sign vitiates the last 20 decimal places in the first calculation of $e^{-\pi/2}$ outlined on p. 432. On the other hand, the second calculation of $e^{-\pi/2}$ when completed yields an approximation correct to 35D.

A few typographical errors appear in this paper. In the definition of A on p. 420 the exponent of 2 should read 7/4 instead of 7/2. On p. 421 the number $340-152\sqrt{5}$ should be equated to the square instead of the fourth power of $\frac{1}{2}(\sin\operatorname{lemn}\frac{1}{4}\tilde{\omega})^4 - \frac{1}{2}(\sin\operatorname{lemn}\frac{1}{4}\tilde{\omega})^4$

On p. 428 the larger prime factor of 32831087 should read 88019 in place of 8819.

It is believed that the preceding enumeration of numerical errors is exhaustive, so that all the data not explicitly mentioned as subject to error may be considered as entirely correct.

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EDITORIAL NOTE: As the result of this communication there will have to be certain revisions in FMR, *Index*, article 5.82.

133. NBSCL, *Tables of the Bessel Functions* $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leqq x \leqq 1$, 1948. See RMT **543**.

The following errata list was issued by the editors soon after publication:

P. vii, l. −6, *for* $\left(\frac{x}{2}\right)^{\nu+2n}$, *read* $\left(\frac{x}{2}\right)^{\nu+2n}$

p. 4, $x = .0096$, $Y_1(x)$, Δ *should read* .683520, *instead of* .652083
p. 11, $x = .0434$, $Y_1(x)$ *should read* −14.720493
p. 13, argument $x = .100$, *not* 100
p. 28, $x = .828$, $Y_1(x)$, Δ^3 *should be* −173
p. 35, $x = .0000$, $K_0(x)$, and $K_1(x)$ *should both read* ∞
p. 42, $x = .099$, $K_1(x)$, Δ *should read* −10198
$x = .100$, $K_1(x)$, Δ *should read* −9997, and Δ^2 *should read* 195
p. 54, $x = .700$, $K_1(x)$, Δ, *for* −2185, *read* −2158.

ARNOLD N. LOWAN

134. TOKIHARU OKAYA, "Note sur le mouvement du fluide parallèle à l'axe d'un cylindre solide et sur le phénomène de Liesegang," *Japanese Jn. Physics*, v. 12, 1937, p. 9–25. See *MTAC*, v. 1, p. 227, 301.

In the table, p. 13, of $\phi_n(x) = (2\pi x)^{\frac{1}{2}}e^{-x}I_n(x)$ there are the following two errors: *for* $\phi_0(0.1^{-1}) = 1.0132970$, *read* 1.0132907; and *for* $\phi_1(0.08^{-1}) = 0.9691305$, *read* 0.9691905. The last decimal in each of the values $\phi_0(0.03^{-1})$ and $\phi_0(0.08^{-1})$ is one unit too high. In equation (15) on p. 12 *for* $\phi_n(x) = \sum_{k=0}^{p}$ *read* $\phi_n(x) = 1 + \sum_{k=1}^{p}$.

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135. J. T. PETERS, *Sechsstellige Tafel der trigonometrischen Funktionen . . . von zehn zu zehn Bogensekunden . . .*, 1929; second ed. 1939; third ed. 1946. See RMT **522**, and *MTAC*, v. 1, p. 121 (MTE **14**), 162 (MTE **27**); v. 2, p. 279 (MTE **103**).

A second edition, reproduced photographically from the first after correction of 22 errors, is reviewed in *Astronomische Nachrichten*, v. 269, p. 287–288 (December, 1939) by "No," presumably Frl. H. Nowacki of the Astronomisches Rechen-Institut, who gives a list of 23 errors in the first edition, including those in MTE **14** and **103**. She refers (see MTE **27**) to the inadequate reading of the leading figures of the first edition, and the subsequent reading of a fresh proof, but does not make it clear whether the list of errors she gives is printed in the second edition or not. She calls attention to the fact that one error out of the 23 (p. 190) has not been corrected in the second edition.

Through Dr. HANNA PETERS, daughter of the author, I have recently received a copy of the third edition (ausgegeben November, 1946); at the request of R.C.A. I have examined this to see if the known errors in the first edition had been corrected. The edition is a photo-

graphic reprint of the first; the cover and list of contents appear to have been reset, but the title page, the "Vorwort" and the remainder of the book appear to have been photographed from the first edition with (a) the addition of "Dritte Auflage" and change of publisher's imprint, (b) the omission of the original date of September, 1928 and (c) the correction of at least 22 errors in the tables. Generally, the reproduction is of a high standard, but occasionally there are minor variations of inking and focus which distinguish it from letter-press printing.

Examination of the Nautical Almanac Office copy of the first edition (in which the 23 errors mentioned above were corrected in early 1940) has revealed further errors which still persist in the third edition; those on p. 49, 269 are in the hand of L.J.C., the others appear to have been found by various members of the staff in the course of use. There is given below a complete list of all known errors in the first edition; those marked with an asterisk * are in the third edition, and presumably in the second edition as well. The error on p. 190 is the uncorrected error in the second edition referred to above. In the first edition a decimal point is omitted from the entry for cos 21°13′00″ on p. 151 and the argument for 66°59′10″ is out of alignment on p. 162; both of these are corrected in the third edition.

Close investigation of a suspected "wrong fount" on p. 55 reveals an amazing mixture of types in this apparently well printed volume. In the main table the figures in front of the decimal point are often in a different type from that generally used. The leading 0 in sin and tan on p. 24 is correct, but most (not all) of the other leading figures are incorrect up to page 81, at which point the mistake was apparently discovered and thereafter corrected; even so the incorrect fount is used in several cases on p. 85. The occurrence is by no means uniform; for instance, it actually spreads to the argument column on pages 32 and 33, while it is correct in the sin and tan columns on p. 43–59. Although the difference between the figures is quite noticeable when compared closely, it is too slight to be seen in normal use and it does not detract from the general appearance of the volume. This mixture occurs in both the first and third editions.

Page	Location	*For*	*Read*
$8^{1,2}$	cotg 0°27′03″, 04″	127 and 126 to be lowered one line.	
12^{1}	diff. cotg 0°40′57″/58″	24	34
48^{1}	heading, last column	sec	cos
$49^{1,2}$	sec 4°10′30″	2261	2661
*49	p.p. table for 86 × 8	68.7	68.8
$55^{1,2}$	argument 5°14′	50″	30″
$61^{1,2}$	sec 6°17′00″	0.	1.
*65	diff. cos 6°55′40″/50″	5	6
84^{1}	argument 79°52′	49″	40″
90^{1}	sec 11°03′20″	9909	8909
110^{1}	argument 75°30′	20″	30″
118^{1}	diff. sec 15°45′40″/50″	44	14
123^{1}	diff. sin 16°38′50″/60″	57	47
127^{1}	diff. tang 17°18′40″/50″	33	53
$129^{1,2}$	sec 17°39′40″	8464	9464
156^{1}	tang 22°06′00″	4.	0.
157^{1}	diff. sin 22°10′00″/10″		45
169^{1}	diff. sec 24°17′00″/10″	4	24
*170	p.p. table for 58 × 9	52.	52.2
*190^{1}	tang 27°44′00″	5.	0.
233^{1}	diff. tang 34°56′00″/10″	22	72
$234^{1,2}$	heading, fourth, fifth cols.	cotg cosec	cosec cotg
261^{1}	diff. sin 39°37′10″/20″	28	38
266^{1}	diff. cotg 40°24′00″/10″	155	115
*269	heading, sin	the s is upside down	
278^{1}	cotg 42°23′00″	0.	1.
291^{1}	diff. sin 44°30′30″/40″	55	35

D. H. SADLER

H. M. Nautical Almanac Office
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[1] These 23 errata were reported in *Astr. Nach.* (1939).—EDITOR.
[2] These 6 errata were reported in *MTAC* (1943 and 1947) by L. J. C.—EDITOR.

136. J. T. PETERS, *Siebenstellige Logarithmen der trigonometrischen Funktionen von 0° bis 90° für jedes Tausendstel des Grades*. Berlin, Preussische Landesaufnahme, 1921.

The following two errors were found during the comparison of this table with proofs of a new CHAMBERS' six-figure table. The comparison covered log sines and log tangents for 0°(0.°001)5°.

P. 74, log tan 3°.619, *for* 8.800 0440, *read* 8.801 0440;
P. 80, log sin 3°.933 *for* 8.836 2102, *read* 8.836 2602.

Neither of these errors occurs in the six-figure or ten-figure tables by Peters of the same functions, published in 1922 and 1919 respectively, and having the same argument.

L. J. C.

137. J. T. PETERS, *Zehnstellige Logarithmentafel*, v. 2, Berlin, 1919.

On p. 762, 38°.000–38°.050, the third digit from the left in the difference column for log tang and log cotg should be 6 instead of 7. Thus for the first difference read 156237, not 157237. This error persists for the entire page.

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UNPUBLISHED MATHEMATICAL TABLES

Reference has been made to an unpublished mathematical table by John Todd in QR35.

69[A].—H. E. SALZER, *Table of Factorials*. Manuscript in possession of the author, NBSCL.

The manuscript is a table of $N!$ for $N = [1(1)1000; 16S]$. Although only 16S are guaranteed, the entries are almost certainly correct to 17S, and there is a high probability that they are good to even 18S, with an error no more than several units in the eighteenth significant figure.

H. E. SALZER

70[E].—G. W. SPENCELEY, *Tables of Hyperbolic Functions*. Manuscript in possession of the author, Miami University, Oxford, Ohio.

I have computed tables of $\sinh x$ and $\cosh x$ for $x = [1°(1°)1080°; 28S]$.

G. W. SPENCELEY

71[I].—H. E. SALZER, *Tables of Coefficients for Checking and Interpolation of Functions Tabulated at Certain Irregular Logarithmic Intervals*. Tables in possession of the author.

Many small tables exist of functions that behave as polynomials in $\log x$. Such functions are usually tabulated for arguments proportional to

1, 2, 5, 10, 20, 50, 100, 200, 500, 1000.

To check such a table for smoothness or to interpolate one uses divided differences. The first of the ms. tables gives the requisite coefficients A for expressing the $(n-1)$-th divided difference of the function $F(x)$ as

$$A_1F(1) + A_2F(2) + A_3F(5) + A_4F(10) + \cdots.$$

The coefficients are given to 8S for $n = 3(1)10$. Two other tables are based on the points

1, 5, 10, 50, 100, 500, 1000 & 1, 2, 10, 20, 100, 200, 1000.

D. H. L.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 418 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

The leading article of this issue of *MTAC*, "The IBM pluggable sequence relay calculator," by Dr. W. J. ECKERT is our current contribution under this heading.

DISCUSSIONS

The second article of this issue of *MTAC*, "Inversion of a matrix of order 38" by Mr. H. F. MITCHELL, JR., is the first of our contributions under this heading. The following five papers are revised summaries of talks delivered at the meeting of the Association for Computing Machinery by members of the staff of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, on December 11–12, 1947; see *MTAC*, v. 3, p. 132–133.

Airflow Problem Planned for the ENIAC

Given a body of revolution with sharp nose immersed in a steady uniform flow at sufficiently high Mach number, it is permissible to neglect viscosity and body forces.

If the independent variables α and β, constant on characteristics, are introduced, it can be shown that the differential equations take the form

$$\begin{aligned} Hy_\alpha - (K+R)x_\alpha &= 0 \\ Hy_\beta - (K-R)x_\beta &= 0 \\ Hu_\alpha + (K-R)v_\alpha + (P+Q)x_\alpha &= 0 \\ Hu_\beta + (K+R)v_\beta + (P-Q)x_\beta &= 0 \\ dz = yA(-vdx + udy) & \end{aligned}$$

where x and y are cylindrical coordinates of a point P, u and v are the corresponding velocity components, and H, K, R, P, Q, A are known functions of y, u, v, z.

The boundary conditions which complete the formulation of the problem are of the form

$$dy/dx = F(x, y) \qquad v/u = F(x, y)$$

on the contour of the body, and

$$G(u, v) = 0 \qquad dy/dx = H(u, v)$$

on the shock-wave.

Approximating the solutions locally by polynomials of first, second, or third degree, one obtains procedures of differing orders of accuracy.

The Airflow Branch of the Ballistic Research Laboratories, with the aid of a group of mathematicians at the Moore School of Electrical Engineering, have programmed for computation by the ENIAC the solution of a 9-parameter family of problems of the above type by a second-order method. The parameters are the Mach number, the coefficients of

two tangent, third-degree curves which make up the head of the projectile, and the coefficients of a third-degree curve defining the boat tail.

The differential equations are nonlinear, hyperbolic, and partial, with boundary conditions on a shock surface not specified in advance. The ENIAC will be able to compute the solution for one set of parameters at 1000 grid points in 20 minutes.

R. F. CLIPPINGER

Nonlinear Parabolic Equations

Numerical solutions of two nonlinear heat-flow problems have been obtained with desk calculators and with large computing machines using finite difference approximations. For both problems the net has been defined by $v_{ij} = v(i\Delta x, j\Delta t)$ and $\Delta t = \frac{1}{2}\Delta x^2$.

In the first problem the system of equations is of the form

$$\begin{aligned} &\partial v/\partial t = \partial^2 v/\partial x^2 + \exp(-1/v), && x \geqslant 0,\ t > 0 \\ &-\partial v/\partial x = H[v_g(t) - v], && x = 0,\ t > 0 \\ &\partial v/\partial x \to 0, && x \to \infty,\ t \geqslant 0 \\ &v = v^{(0)}, && x \geqslant 0,\ t = 0. \end{aligned}$$

The approximately equivalent difference equations

$$\begin{aligned} v_{i,j+1} &= \tfrac{1}{2}(v_{i-1,j} + v_{i+1,j}) + \Delta t \exp(-1/v_{i,j}), \qquad i > 0 \\ v_{0,j+1} &= v_{1,j} + H\Delta x(v_{gj} - v_{0,j}) + \Delta t \exp(-1/v_{0,j}) \end{aligned}$$

were solved by hand calculation and with the Bell Relay Computer for three values of H, four values of $v^{(0)}$, and various simple functions $v_g(t)$. It appears that the local truncation error committed at the boundary is greater than in the interior and that attainment of significantly smaller error may entail excessively laborious computation. The integrated error is difficult to estimate because of the nonlinearity. However, completed runs with several mesh sizes on the Bell machine may yield, at some later date, a better than qualitative estimate. An approximate analytic solution has been obtained which permits estimation of the t for which $v(0, t) = 0.05$ to an apparent accuracy of 20 percent in t over a range of 10^9. The performance of the Bell machine on this problem has been quite satisfactory. No errors have been detected in its 1000 hours of operation, about 50 percent of which were unattended.

The second problem requires the solution of the system

$$\begin{aligned} &\partial v/\partial t = \partial^2 v/\partial x^2 + m\mu(t)(\partial v/\partial x), && x > 0,\ t > 0 \\ &v = f(x), && x \geqslant 0,\ t = 0 \\ &v = f(0), && x = 0,\ t \geqslant 0 \\ &v \to 0, && x \to \infty,\ t \geqslant 0 \\ &\mu(t) = 1 + 2\partial v/\partial x, && x = 0,\ t \geqslant 0 \end{aligned}$$

where $f(x)$ is a given function, m is a parameter, and $\mu(t)$ is to be determined.

Numerical integration has been performed on both the IBM Relay Multipliers and the Bell Relay Computer, using the simplest finite difference approximation for the differential equation

$$v_{i,j} = \tfrac{1}{2}(1 - \tfrac{1}{2}m\Delta x\mu_j)v_{i-1,j} + \tfrac{1}{2}(1 + \tfrac{1}{2}m\Delta x\mu_j)v_{i+1,j}$$

and a third order forward difference formula for

$$\mu_j = 1 + \Delta x^{-1}[-\tfrac{1}{6}f(0) + 6v_{1,j} - 3v_{2,j} + \tfrac{1}{3}v_{3,j}].$$

Computation proceeds along "lines," the values for fixed j and all i being computed before going on to $j + 1$. It is necessary to start with a small Δx and to double it several times to cover the desired range in t.

The computation on the Bell Computer was relatively slow, about 50 seconds being required for each point. However, the operation was completely automatic; the machine ran unattended overnight several times. No machine errors were found.

The IBM Relay Multiplier calculates more rapidly, requiring only 1.2 seconds for each point, but the operation is not completely automatic. After each line the cards must be transferred by hand and the coefficients in (1) set on switches. Also, to check against machine errors, it is necessary to make a run of each line on a second Multiplier and compare $\sum_i v_{i,j}$. The integral of $\mu(t)$ is also required and is computed automatically on the Bell Computer but must be done separately by hand from IBM results. For these reasons, the machine operating time alone does not determine the total time required for solutions.

Accuracy on the IBM Multiplier is severely limited because the machine operates with only 6D digits. When the interval size was reduced to cut down truncation errors, the rounding off errors became serious.

In conclusion, it should be mentioned that experience with solution of these nonlinear parabolic equations is in accordance with the experience of others, namely: that fast and long calculations on automatic or semi-automatic computing machines require a proportionately large amount of mathematical analysis, often complex in nature.

BRUCE L. HICKS & H. G. LANDAU

Laminar Boundary Layer Flow in a Compressible Fluid

A problem recently placed on the ENIAC involves the solution of the partial differential equations representing laminar boundary layer flow in a compressible fluid. The particular problem is for the case of a flat plate with the flow at zero incidence, assuming a constant but unspecified pressure gradient. The solution is obtained by series expansions in one of the independent variables with coefficients being functions of the other independent variable. The coefficients are obtained by the solution of a set of ordinary differential equations. A set of ordinary differential equations for each order of coefficients is obtained by substituting the series expansions in the original partial differential equations and equating coefficients of the independent variable of the series expansions.

In this application the great advantage of using a high-speed machine such as the ENIAC arises from the fact that the solution is obtained by meeting two boundary conditions at $Y = \infty$. This involves guessing two boundary conditions at $Y = 0$ and integrating the equations to find out if the boundary conditions at infinity are met. (In this problem ∞ is approximately 250 intervals of the independent variable Y.) It usually requires several runs to obtain an agreement with the boundary conditions specified at $Y = \infty$, and each integration is a sizable computing problem by hand. A machine with a larger memory would be more efficient than the present ENIAC since this would eliminate the printing of many intermediate quantities, which is necessary with the present machine.

This method of solution was proposed by Professor D. R. HARTREE of Cambridge University, England.

JOHN V. HOLBERTON

On the Approximate Solution of a Partial Differential Equation on the Differential Analyzer

An attempt was made to obtain an approximate solution on the Differential Analyzer for the partial differential equation $v_t = v_{xx}$, subject to boundary conditions $v = v_0(x)$ at $t = 0$ where $v_0 \to 0$ as $x \to \infty$, and $v =$ constant at $x = 0$.

The scheme tried consisted of replacing the above equation by the second-order ordinary differential equation

$$d^2[v_{j+1}(x) + v_j(x)]/dx^2 = 2[v_{j+1}(x) - v_j(x)]/\Delta t$$

where $v_j(x) \equiv v(x, j\Delta t)$, $j = 0, 1, 2, \cdots$. Fixing Δt and beginning with $j = 0$, an operator manually feeds in the function $v_0(x)$ from a graphical input table, obtaining $v_1(x)$ as a graphical solution. Then, feeding in $v_1(x)$ on the second run, the solution obtained is $v_2(x)$, etc. The solution desired on each run is that solution which approaches zero asymptotically.

It was found that the solution of the equation on the Analyzer is highly sensitive to errors in feeding the input function, and at the slightest provocation the output curve proceeds to diverge at a wild rate from the solution sought. An explanation of this behavior is found by rewriting the foregoing ordinary differential equation in the form

$$d^2[v_{j+1}(x) - v_j(x)]/dx^2 - 2[v_{j+1}(x) - v_j(x)]/\Delta t = -2d^2v_j(x)/dx^2.$$

Since v = constant at $x = 0$, the general solution of this differential equation is $V_{j+1} = v_{j+1} + c \sinh (2/\Delta t)^{\frac{1}{2}} x$, where v_{j+1} is the particular solution satisfying the boundary condition at infinity, and c is an arbitrary constant. The second term on the right, which is the solution of the homogeneous equation corresponding to the differential equation, will, for nonzero values of c, cause V_{j+1} to diverge to $\pm\infty$. A nonzero value of c sufficiently large to produce very rapid divergence can be introduced by a very minute error in manual feeding of the input function. In the case $j = 0$, for example, it was found that even with the most painstaking following of the input v_0 not only is it impossible to obtain a solution V_1 which approaches zero, but in successive runs with the same initial and boundary conditions it is impossible to obtain the same V_1 twice. The worst divergence from the desired V_1 which occurred was caused by an error in manual following of 0.017 in. in the vertical scale over an interval of 0.5 in. in the horizontal scale. Since it is virtually impossible on the Aberdeen Analyzer manually to follow a graph with an accuracy of more than .02 inch, it is easy to see in a problem of this sort, and with the setup used, that only the remotest chance could yield a solution which satisfies the boundary condition at infinity. For the above reasons, the Analyzer approach to the solution of the partial differential equation given had to be abandoned.

JOSEPH H. LEVIN

Computation of the Airflow about a Cone Cylinder

In the past, airflow computations had to be performed with the aid of standard desk machines only. They will in the future be carried out by high-speed machines such as the ENIAC. In order to convey some idea of the complexity of this type of computation, a brief description is given here of a method now used for the determination of the supersonic flow of a compressible gas about a nonyawing cone cylinder placed in a uniform stream. Assuming steady and irrotational flow, the equations governing the flow are found to be

$$Hu_x + K(u_r + v_x) + Lv_r + P = 0, \qquad v_x - u_r = 0.$$

In these equations, (x, r) are Cartesian coordinates of a point in a fixed meridian plane, in the direction of the axis of the body and normal to it, respectively; u, v are the components of the velocity vector in the x, r directions; u_x is the partial derivative of u with respect to x, etc.; $a^2 = \frac{1}{2}(g-1)(1-u^2-v^2)$ is the square of the local speed of sound, measured in units of the speed of efflux into a vacuum; $g = c_p/c_v$ is the ratio of the specific heats of the gas; and $H = a^2 - u^2$, $K = -uv$, $L = a^2 - v^2$, $P = a^2v/r$. The above system of partial differential equations is hyperbolic if the flow is supersonic. In this case it is convenient to introduce another pair of variables, called characteristic variables, and reduce the system to an equivalent but simpler system of differential equations.

The computation then proceeds as follows: Given the free stream velocity q_1 and the semi-cone angle h, the location (x, r) of the characteristic SW off the shoulder S and the velocities u, v along SW may be obtained by integration of ordinary differential equations.

Integration of another ordinary differential equation is necessary to rotate q_1 at S through the angle h until it becomes parallel to the cylindrical surface. Now the flow in the next region may be computed; here the characteristic differential equations are advantageously replaced by their equivalent difference equations; these permit computing x, r, u, v at P, once these quantities are known at the adjacent points below and to the left of P. The computation then proceeds to fill out subsequent regions.

From the results thus far obtained it would seem that even relatively coarse characteristic grids—which may be completed fairly quickly—lead to physically meaningful numbers.

Thus, good results have been obtained in many cases where characteristic grids consisting of approximately 120 points were used. Since it takes an experienced computer working with a standard desk machine up to 45 minutes to do one point of such a grid, a job of this nature requires roughly 90 hours.

When higher accuracy is desired, however, it frequently becomes necessary to work with much finer grids and, consequently, to perform an amount of work often larger by a factor of ten than that mentioned above. It is in cases such as these that high-speed computing machines, like the ENIAC, become most essential, especially if a large number of such large-scale calculations of the same type have to be carried out. But even the ENIAC, which was originally designed to do mainly firing-table work, has been found to lack the capacity needed for such extensive flow problems. Once this defect has been remedied, it is hoped that this machine, working at its best about 5000 times faster than man, will also be well suited for flow calculations.

M. LOTKIN

BIBLIOGRAPHY Z–IV

1. HOWARD H. AIKEN & GRACE M. HOPPER, "The automatic sequence controlled calculator—I," *Electrical Engineering*, v. 65, 1946, p. 384–391, 6 figures. 21.6 × 27.9 cm.

Beginning with a short resumé of the major stages of computing machine development from the abacus through Charles Babbage's Analytical Engine, this article continues with a short discussion of the increased need for mechanical computation and goes on to describe the sequence control unit of the automatic sequence controlled calculator, the storage counter, and the "ganged counters." A bibliography of 21 items is appended.

With the increasing need for accurate computation occasioned by the use of numerical analysis in many scientific developments, as well as by greater accuracy of physical measurement, more and more time and effort must be devoted to computational labor, which is, moreover, always susceptible to human fallibility. However, the situation has improved somewhat with the development of the automatic sequence controlled calculator, which will carry out any selected sequence of the 5 fundamental operations of arithmetic (addition, subtraction, multiplication, division, and reference to tables of previously computed results) under completely automatic control. This article, Part I of 3 parts (see *MTAC*, v. 2, p. 316), describes the mechanism of the machine and explains its function in addition and subtraction.

MDL

2. ANON., "Electronic digital computers," *Mechanical Engineering*, v. 69, May, 1947, p. 413–414. 28.6 × 20.9 cm.

3. ANON., "Mark II Calculator," *Rev. Sci. Instrs.*, v. 18, March, 1947, p. 202. 26.7 × 19.7 cm.

4. E. C. BERKELEY, "Electronic machinery for handling information, and its uses in insurance," Actuarial Soc. Amer., *Trans.*, v. 48, part 1, 1947, p. 36–52. 22.9 × 15.2 cm.

This paper is divided into 4 sections. The first gives a concise description of 4 high-speed machines already in use: the Differential Analyzer at MIT, the Harvard Sequence Controlled Calculator, the ENIAC at Aberdeen, and the Bell Telephone Laboratories' Relay Calculator. In each case the author supplies information regarding memory capacity, storage medium, rapidity of access, types and speeds of operations performed, and reliability of results. He points out an especially attractive feature of the BTL Machine—its built-in checking circuit which stops the machine when a discrepancy is indicated. The performance of these machines proves, according to the author, that high-speed calculators are entirely successful, that great complexity of computational routine does not affect their efficiency, and that

they are economically feasible. For, even at their present high cost—between one-fourth and one-half million dollars—the savings to an owner would be about 6 or 7 times the price if one assumes only a 10-year life span for a machine. Obviously, when a considerably cheaper standard model is manufactured, the savings will be far greater.

The second section deals with experiments in the field of electronic machine construction. The problem of devising an efficient and inexpensive medium for storage of information seems to have been solved by the use of magnetic wire or tape, which can store about 2×10^9 binary digits per cu. ft., retain the information indefinitely, and accept new information in place of old, the latter being erased as it is "written" over.

Experiments also show that extraordinarily high speeds may be achieved on the proposed electronic machines for the mathematical operations of addition, subtraction, multiplication, and division, as well as for the logical operations of comparing, selecting, sequencing, etc. Thus, electronic circuits are being developed that will perform up to 100,000 additions per second, and multiplication may reach a speed of 10,000 per second. Mr. Berkeley points out, however, that in the insurance business the need for such enormous speeds is at present not pressing.

The paper continues with a discussion regarding the programming of a problem for an electronic machine. All the instructions, as well as the data, will be stored in the memory, so that the entire problem can be handled by the machine without human intervention. The instructions will be such as to enable the machine to modify the sequence of operations automatically, depending on intermediate results obtained during the course of the computation.

Much thought has, of course, been given to making these machines highly reliable as regards accuracy of final results, infrequency of machine failure, rapidity of localizing the seat of failure when it does occur, ease of repair, and prolonged operation without the presence of human attendants. As might be expected, the new machines will not only be speedier and more trouble-proof than the first experimental models, but cheaper as well, their cost being estimated as between $100,000 and $125,000.

In the third section, the author lists the features of an electronic machine which would be ideal for performing the chores of a large insurance company. This machine would have a large memory capable of rapid reference, as well as ready access to about 200 tables aggregating some 100,000 ten-digit numbers. It would maintain accurate up-to-date files and take proper action based upon these data according to insurance company rules. As regards the input apparatus, Mr. Berkeley would like to have data converted for machine use from punch cards, punched tape, or a typewriter with equal ease. Since manual operations are exceedingly slow, it is obvious that multiple-input devices will be needed to keep up with the great speed of the electromatic machines. He also expects highly reliable performance, as well as a host of other things, hoping at the same time that the cost can be kept within reasonable limits. (It is the pleasant task of the MDL to report to the readers of MTAC that all these expectations are not far from realization.)

The last section gives a fairly complete listing of the various insurance problems which the electronic machine is expected to solve with great efficiency. As the paper under review is but a chapter of a forthcoming book, there is no need to go into greater detail here. The author points out one great advantage that these machines will bestow upon both clerks and actuaries of an insurance company: it will relieve them of much drudgery and enable them to use their time and abilities in fascinating investigation and research which lack of time has hitherto prevented.

MDL

5. ARTHUR W. BURKS, "Super electronic computing machine," *Electronic Industries*, v. 5, no. 7, p. 62–67, 96, 9 figures, 4 illustrs. 21.6 × 29.2 cm.

6. L. J. COMRIE, "Calculations and electronics, automatic computer designed by the National Physical Laboratory," *Electrician*, v. 137, Nov. 8, 1946, p. 1279–1280. 14 × 22.2 cm.

7. E. U. CONDON, "Electronics and the future," *Electrical Engineering*, v. 66, 1947, p. 355–356. 29.2 × 22.2 cm.

8. HASKELL B. CURRY & WILLA A. WYATT, *A Study of Inverse Interpolation on the ENIAC*, Ballistic Research Laboratories, Report No. 615, August, 1946, 58 pages, 50 figures. 27.9 × 21.6 cm.

Given $x = x(t, \phi)$ and $y = y(t, \phi)$ in tabulated form for equally spaced values of t and ϕ, the problem is to express t and ϕ in tabular form for equally spaced values of x and y. The purpose of the report is to provide a basic program for solving this problem on the ENIAC in connection with firing table computation.

Although the problem is one of interpolation in two variables, it is treated here as successive inverse interpolation on one variable at a time. The inverse interpolation is done by an iteration procedure since, the authors say, "it is eminently suitable for the ENIAC," and "the process is in principle independent of the choice of interpolatory approximation formula."

The chapter headings are Introduction, General Outline of the Calculation, Theory of the Iteration, Methods of Discriminating, Theoretical Considerations Related to Formulation of Linear Expressions, Schedule of Stages for the Basic Scheme, Detailed Program for the Basic Scheme, Modifications Near a Maximum, A Second Method of Successive Approximations, Composite Interpolation (interpolating for several functions simultaneously), and Concluding Remarks. A number of diagrams illustrate the interpolation process for various cases; others give the explicit ENIAC coding for the basic scheme and for various modifications.

MDL

9. J. S. KOEHLER, "An electronic differential analyzer," *Jn. Appl. Physics*, v. 19, 1948, p. 148–155. 26.7 × 19 cm.

"An electrical device is designed which will solve ordinary, nonlinear, nonhomogeneous, differential equations or the boundary value problems based on such differential equations. The device is based on the fact that the charge on the condenser in a series resonant circuit varies with the time in a manner described by a linear second-order differential equation. The desired variations of the coefficients with time are fed into the apparatus by using a variable voltage generator. This generator provides a small voltage which varies with time in accordance with any given curve. The variable voltage generator has been constructed and operates satisfactorily. The resulting solution is drawn by an oscillograph. The instrument is designed to furnish a solution which does not deviate from the true solution by more than 4%."

10. F. RUSSO, "Les grandes machines mathématiques," *Revue des Questions Scientifiques*, s. 5, v. 8, 1947, p. 611–616. 25.4 × 16.5 cm.

Summaries of talks delivered by Professor Léon Brillouin of Harvard University, and Louis Couffignal, in charge of mechanical computer studies at the Centre National de la Recherche Scientifique, at a meeting at the École des Telecommunications in Paris on June 12 and 13, 1947. The talks are specifically concerned with recent developments in computing and the need in France for large-scale computers. A project for the development of a large-scale electronic computer will be under way at the beginning of 1948, and it is hoped that the machine will be operating in 1949.

MDL

11. R. R. SEEBER, "Value of super-calculators," *National Underwriter* (Life Edition), no. 4, April 4, 1947, p. 6. 25.4 × 33 cm.

In an address to the junior branch of the Actuaries' Club of New York, Mr. Seeber stresses the possibilities for use of the "super-calculators" in certain types of actuarial calculations.

MDL

12. S. LILLEY, "ENIAC, ASCC and ACE, machines that solve complex mathematical problems," *Discovery*, v. 8, January, 1947, p. 23–27, 32. 25.4 × 18.4 cm.

13. Tjänste, Chalmers Tekniska Högskola, Göteborg, *Betankände med förslag till närmast erforderliga åtgärder för tillgodoseende av Sveriges behov av matematikmaskiner*, April 30, 1947, mimeographed. 36 leaves, 7 illustrs. 21 × 29.8 cm.

This report contains a proposal presented to the Secretary and Chief of the Royal Ecclesiastic Department by the Mathematical Machine Committee of Sweden for the acquisition of one or more of the new high-speed computers for Swedish use. It discusses the various developments in high-speed computation, a field in which Sweden has, up to the present time, done very little. In view of the urgent need of the Swedish Armed Forces and industries, it was concluded that an electronic computer should be immediately acquired from abroad, if possible, after a thorough study by Swedish experts of the projected foreign machines, and that these experts should familiarize themselves with computing developments and techniques in foreign countries. The price of the machine is not to exceed 2,000,000 kronor, and 100,000 kronor is to be allocated as a reservation estimate for the budget year 1947–1948. The Committee has also been instrumental in the establishment of an organization to undertake the concrete planning and action necessary for the purchase of a large-scale computer.

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14. ROBERT C. TUMBLESON, "Calculating machines," *Federal Science Progress*, v. 1, June, 1947, p. 3–7, 12 tables. 29.2 × 22.9 cm.

Special mention is made of the growth, commercial applications, and projected developments in the field of automatic computing machinery. The author describes the essential features of the Automatic Sequence Controlled Calculator and the ENIAC and continues with a discussion of some of the big computer projects now under way, namely, the EDVAC, the computer under design at the Institute for Advanced Study, and the computers being designed for the Bureau of Census and the Office of Naval Research under the direction of the National Bureau of Standards. The far reaching significance of developments in this field is stressed. "For the first time, technical men are freed from the enormous drudgery of calculations. At last they are free to think."

MDL

NEWS

Electronic Digital Computing in England.—There are four digital computing projects under way in England. One of these is located at the National Physical Laboratories at Teddington, Middlesex, another at the Mathematics Laboratories of Cambridge University, and a third at Manchester University. The fourth is under the sponsorship of the British Rubber Industries.

The work at the National Physical Laboratories (NPL) was started in the Mathematics Division under the direction of Dr. A. M. TURING. Until January of last year the efforts of his staff, consisting of one full-time and one half-time assistant, were concentrated on coding and logical design while the development work was let as a contract to the Post

Office Research Station at Dollis Hill. However, since the responsibilities of the Post Office group primarily involved communications, the work for NPL did not proceed very fast. This led part of the group at Teddington to begin the design and development work necessary to construct a pilot model. At this stage 6 people were doing this design and development as well as carrying on logical and coding investigations relating to the design of a large Automatic Computing Engine,[1] or ACE, as they called it.

In July 1947, Sir CHARLES DARWIN, Director of the NPL, decided to expand the program for developing and constructing the computing engine. An immediate result was the formation of a group of about 15 persons in the Radio Division. Some of the personnel working on the pilot model were assimilated by this new group, and their supplies and equipment were taken over. However, there remained in the Mathematics Division about 5 persons who were to work on coding. At this time Dr. Turing, having requested and received a year's leave of absence, returned to Cambridge to resume a fellowship there.

The group in the Radio Division was officially established on August 15 with personnel mainly borrowed from other divisions of the laboratory. None of these people had any previous experience with computing machines, and most of them had little experience working at the pulse frequencies of such machines. As they had very little supplies and equipment, most of their time up to January 1948 was spent in obtaining supplies and equipment and in converting a room for use as a laboratory.

The development and construction of a mercury delay line computing machine was begun at Cambridge University by Dr. M. V. WILKES. By means of a grant from the Department of Scientific and Industrial Research,[2] Dr. Wilkes was able to spend some time in the United States; in particular, he was able to attend some of the lectures on computing machines, given at the Moore School of Electrical Engineering at the University of Pennsylvania in Philadelphia during the summer of 1946. During 1947 he produced a circulation system in a mercury delay line operating at a 500 kilocycle pulse frequency. With the help of at first one and later two men, he then proceeded to design, construct, and test other components of his computing machine.

Professor F. C. WILLIAMS studied the problem of storing pulses on the face of a standard cathode-ray tube while still at Telecommunications Research Establishment at Malvern. In 1947 he moved to Manchester University to teach in the electrotechnics department and there continued his research on the storage problem. Professor M. H. A. NEWMAN of the mathematics department of Manchester University is strongly interested in the development of a high-speed computing machine.

At the end of 1947, Professor Williams was so successful in storing 2048 pulses on the face of a twelve-inch tube that he hoped to be able to store 4096 pulses by essentially the same technique. In order to do this, he thought the tube might need a more spherical face to simplify the focus problem, a smaller gun aperture to produce smaller spots on the tube face, and possibly a metallic collector ring to avoid carbon particles in the fluorescent material on the face.

He hopes to build what he refers to as a baby machine by about July 1948. This machine will contain only a few storage tubes, rudimentary computing circuits, and probably a manual (keyboard) input and a visual output.

Mr. A. D. BOOTH has been working on automatic computing machines for some time under the sponsorship of the British Rubber Products Research Association. This work is being done at Welwyn Garden City, Hertfordshire. Although his first machines used relays, more recently he has used magnetic paper discs for the memory. Initially, this magnetic-disc machine used relays in the auxiliary circuits, but it is understood that the relays were to be replaced by "valves." Mr. Booth has been handicapped by the fact that there has been practically no development of magnetic tape or wire equipment in Britain. He obtained his paper discs in the United States and has had to wind his own recording and reading heads. The latest memory device used in this machine is a rotating drum on which magnetic impulses representing the data are impressed.

Returning our attention to the logical designs developed at the Mathematics Division of the NPL, we find these designs influenced by Dr. Turing. The machine has many logical

facilities,[3] and as Mr. Todd [4] says, arithmetic facilities were added only as a concession to those interested in computing. The approach to the design has been from the viewpoint of a *universal* computing machine.[5] This is illustrated by the general principle of having as simple circuits as possible outside of the memory and doing the more complicated activities by relatively elaborate programs (sequences of instructions in the memory).

Emphasis on a simple machine led to more complicated systems of coding than were used in any of the computers being designed in this country. Thus, one instruction tells from what delay lines two numbers are to be taken, what operation is to be performed upon them, into which delay line the result is to be placed, and from which delay line the next instruction is to come. The actual words taken from the respective delay lines are determined by a timing number. The next instruction is taken from the memory as soon as the present instruction is obeyed. Thus, instructions to be obeyed successively must be scattered along in the particular delay line; for example, successive instructions may be in positions 1, 7, 29, 6, 13, 3, ... This appears to be a considerable complication, but those at NPL who are familiar with the system do not think so. They find it relatively easy to fill about 80 percent of a delay line with instructions before it is necessary to shift to a new one. Furthermore, it is not difficult, while shifting to new lines, to refer back and fill in any remaining gaps in the delay lines.

The following is a comparison of the pilot model and the ACE being designed at the NPL, the machine at Cambridge University, and the small machine being planned at Manchester University.

The memory of the pilot test assembly planned at the NPL was about 250 10-digit numbers and was to be realized in mercury delay lines about 5 feet long, with a pulse repetition frequency of one megacycle and using a carrier in the delay line of fifteen megacycles. The ACE has always been planned with a memory of 4000 to 6000 10-digit numbers, consisting either of delay lines similar to those in the test assembly or storage tubes of the type developed at Manchester if they were definitely successful. The memory of the Cambridge machine is about 512 10-digit numbers and is realized in mercury delay lines working at a pulse repetition frequency of 500 kilocycles. The memory of the Manchester machine is to be a "few" cathode-ray tubes. The effective pulse frequency in this machine is about 125 kilocycles.

At the NPL, Hollerith (IBM) equipment was to be used for input and output, and the possibility was held open for using magnetic wire or tape on the ACE when it was built. The Cambridge machine is to use teletype tape. At Manchester no development work has been done on input-output mechanisms; it is hoped that magnetic wire or tape equipment may be obtained from someone in the United States.

The word length ranges from 32 pulses, for the test assembly (at NPL) and the Manchester machine, to 36 pulses for the Cambridge machine, and 40 for the ACE. All machines are purely binary in character with conversion to be done by programming. The Cambridge and Manchester machines make use of a single address code very similar to that used at Princeton. As explained above, the NPL machines use a coding system quite distinct from any other.

Expected completion dates of the various machines may be of some interest. The test assembly at NPL may be finished this year, and it is hoped that the ACE will be completed in about three years. Dr. Wilkes expects to have the Cambridge machine in operation by the end of this year. The "baby" machine at Manchester should be ready for use sometime this summer, but no dates are quoted there concerning any full-sized machines.

HARRY D. HUSKEY

NBSMDL

[1] In England "engine" has been a favored term for automatic computing machines since the days of CHARLES BABBAGE and his "analytical engine."

[2] The NPL come under this department, although, due to their early history, they are not administered in the same manner as other subdivisions of D.S.I.R.

[3] For example, an "and" facility which gives a result only if the numbers stored in two locations in the memory agree in some of their digits.

[4] Mr. JOHN TODD of the University of London is spending some time with the Institute for Numerical Analysis of the National Applied Mathematics Laboratory.

[5] See A. M. TURING, "On computable numbers, with an application to the entscheidungsproblem," London Math. Soc., *Proc.*, s. 2, v. 42, p. 230–265, 1936.

Automatic Relay Computer (ARC)—Supplementary Notes.—The ARC, part relay, part electronic, calculating machine, referred to above, is now nearing completion (at the laboratories of the British Rubber Producers' Research Association), under the direction of Dr. A. D. BOOTH and Miss K. H. V. BRITTEN, both of King's College, London.

The logical design of the ARC, which was developed by Booth and Britten, is described in a report prepared by them at the Institute for Advanced Study, Princeton, N. J., in September, 1947. The coding is similar in principle to that proposed by VON NEUMANN and GOLDSTINE.

This machine is a completely automatic computer, capable of dealing with any problem which can be expressed in terms of the processes of ordinary arithmetic (addition, subtraction, multiplication, and division). It is designed to handle numbers of 20 dyadic digits.

The ARC comprises an arithmetic unit and control, both constructed of standard telephone relays, and a memory consisting of a rotating drum on which data are impressed in the form of magnetic impulses. The binary scale is used, and the machine is constructed primarily to work to 6D accuracy though more places can be retained if desired.

Problems are prepared for the machine by translating them in terms of its "code." This is simply the list of orders which it is possible for the machine to obey and consists of 26 orders of the type "Add number from position (x) of memory into accumulator" and "Multiply number in position (x) of memory by number in shifting register." When a problem has been "coded," the sequence of orders and any necessary initial data are punched onto teletype tape and transferred from there to the memory. Results are also recorded on tape.

The main bulk of construction is now complete, and the machine has been in action as a multiplier, multiplying two 20-binary digit numbers in 1/3 sec. A dividing unit which is expected to work at the same speed is nearly complete.

A note on size may be of interest. The arithmetic unit and control consist of about 600 relays, occupying two racks 5 ft. by 2 ft. The actual memorizing part of the memory will be 2 in. in radius and 14 in. long, and the accompanying electronic circuits will not contain more than 120 tubes. Thus, the whole machine, including input and output, will occupy a floor space of approximately 6 ft. by 3 ft.

The American Academy of Arts and Sciences.—On Wednesday, February 11, 1948, at Boston, Massachusetts, a meeting was held at which Professor JOHN VON NEUMANN of the Institute for Advanced Study, Princeton, N. J., spoke on "Electronic methods of computation." Professor von Neumann showed that fast computing, at electronic speeds, requires complete automatization of a large area in logic and extensive foresight concerning its possible ramifications, functions, and malfunctions. It promises, therefore, to initiate a new, quantitative branch of logic. It also provides a new approach to nonlinear problems, which now block many important avenues in mathematics and mathematical physics. Subsequent applications will be still broader.

Association for Computing Machinery.—Since the Association now contains over 350 paid-up members in all parts of the United States and in some foreign countries, the Executive Committee, at a meeting in New York on January 16, dropped the word "Eastern" and changed the name of the Association to the Association for Computing Machinery. A proposed constitution and bylaws have been drafted, submitted to the Executive Council for discussion, and revised. This new draft was sent to the members by mail for balloting.

IBM Selective Sequence Electronic Calculator.—The new IBM Selective Sequence Electronic Calculator was dedicated to the use of science by THOMAS J. WATSON, President of the Company, in a series of elaborate ceremonies lasting through January 27 and 28.

The dedication took place at the Company's New York Headquarters, 590 Madison Avenue, in the presence of some 200 representatives of science and industry, to whom the existence of the machine came as a surprise; that an undertaking of such magnitude could be carried out for two years in complete secrecy is a tribute to the numerous members of the IBM organization who were involved in the plan.

Within its huge glass-and-steel room, the giant SSEC—containing 12,500 electronic tubes, 21,400 relays, and 40,000 pluggable connections—was computing positions of the moon from a lengthy formula, representing a function of time at 6-hour intervals. Each computation was completed and checked in 7 minutes, although it involved no less than 10,710 additions and subtractions, 8,680 multiplications, and 1,870 references to a table of sines. This impressive speed is due to several factors, of which the most important are

1. The ability of the machine to read, modify, and execute instructions stored in the same manner as numbers.
2. The 8 high-speed busses (each consisting of 78 channels) which connect the electronic memory with the arithmetic unit, allowing a speed of 3,500 additions, 50 multiplications and 30 divisions per second.
3. The 36 table look-up units which make it possible to carry out a complete search in a table containing a maximum of 100,000 decimal digits in, at most, 3 seconds.

Scientists interested in the services of the SSEC are advised to apply to Dr. W. J. Eckert, Director of the Watson Scientific Computing Laboratories at Columbia University or, preferably, to their local IBM office.

OTHER AIDS TO COMPUTATION

A New Class of Computing Aids

Students of computing aids have been accustomed to putting computers into one of two classes. One class includes "continuous" devices that represent mathematical quantities by measurements of some analogous, continuously variable physical quantity, like length, voltage, angle, etc. The accuracy of such computing aids is strictly limited by the errors in the physical measurements. The second class consists of digital computers that represent mathematical quantities first in a digital or radix notation, such as the decimal or the binary notation; they then represent each digit of this notation by setting up certain discrete physical situations, like 10 stable positions of a counter wheel or the off-or-on conditions of an electric switch. The "capacity" of a digital machine is the number of digit combinations it can handle, and is unrelated to errors of measurement or of construction provided these remain within certain broad limits.

I would like to suggest that a recently developed computing element makes a third category desirable, for it does not belong to either of those mentioned. The "function unit" computer represents numbers by a counting process instead of a measurement. Since counting is an exact process, the capacity of a function unit is limited only by the number of counts it can make and not by mechanical precision. On the other hand, the function unit does not represent quantities in a digital notation, so it is not a digital computer.

The name I suggest for the third class of computers is *counting computers*.

Some later remarks on the properties of counting computers will be more easily understood with a short description of a member of the class as back-

ground. The "function unit" is a mechanical representation of a function of one variable. Typically, the input or argument variable, x, is introduced into the function unit by rotating a shaft, and counting the revolutions of that shaft. Similarly, the output or function, y, is supplied by the function unit in the form of rotations of an output shaft.

The relation between y and x may be written

$$y = f(x).$$

It is stored in the form of a "tape" record, that is "read" mechanically by the function unit. Several modes of recording and reading have been devised, but the essential features are, first, that the tape is divided into recognizable steps that can be counted; second, that the record instructs the output shaft to rotate in countable steps; and third that the tape or record can be made to have as many steps as may be desired, by making it long enough. In brief, the function unit says that y is a function of x and any particular tape inserted in the unit specifies the particular values of the integers y that belong to the integral values of x over some extensive range of values of x.

Other elements adaptable to use in a counting computer are gear trains and the special combination of gears called "adders," or in engineering language, "differentials." Since gear teeth mesh positively in a one-to-one relation, and since the number of revolutions that gears can make is unlimited, these are suitable components for a counting computer. They supply us with means for adding, and for multiplying by a constant. Thus we can mechanize three kinds of mathematical relations in a counting computer:

$$\begin{array}{ll} y = f(x) & \text{(function unit)} \\ w = u + v & \text{(adder)} \\ w = ku & \text{(gear train).} \end{array}$$

In these relations, x and y are essentially integers (some forms of function unit combine the counting principle with a continuous subdivision of the integers), u, v and w are real numbers and k is a fixed rational number that cannot be changed without changing the gears.

In contrast, the desk calculator—a digital machine—performs the operations

$$w = u + v, \qquad w = u - v, \qquad z = u/v,$$

where u, v and w are integers and z is rational. Continuous computers perform a multitude of operations which may include

$$z = \int x\,dy, \qquad z = dy/dx, \qquad z = \sin x, \text{ etc.}$$

While, at first glance, it may seem that the operations mechanized by the counting computer are too restricted to be useful, further consideration shows that it is possible to combine them in many ways. Many functions of two or more variables can be expressed as linear combinations of functions of single variables, where the latter are, in turn, linear combinations of the original variables. Thus a frequently used function of two variables is the product,

$$z = xy.$$

We can form new variables u and v where

$$u = x + y, \qquad v = x - y,$$

by the use of adders. The shafts representing u and v drive function units with square-law tapes, whose outputs are t and w, where

$$t = f(u) = u^2, \qquad w = f(v) = -v^2.$$

The outputs of these function units enter an adder which forms

$$t + w = (x + y)^2 - (x - y)^2 = 4xy.$$

Finally a gear with a ratio $k = 1/4$ leads to

$$z = 1/4(4xy) = xy.$$

Continuous functions, like $\sin u$, $\log u$, or arbitrary functions are easily approximated in the function unit, since the variable u can be multiplied by a very large constant, k, such as $k = 5000$, to give us a new variable, x. When u ranges from $-\pi$ to π, for example, x goes from $-15{,}708$ to $+15{,}708$. Similarly we can multiply $\sin u$ by a constant, say 5000 again, and let $f(x)$ be the integer nearest to

$$5000 \sin (x/5000),$$

where x is the integer nearest to $5000\,u$. Then the error in approximating u is no greater than half of $1/5000$ or .0001, and the error in the function is no more than an additional .0001. The probable error is, of course, much less.

Similarly, the function $\log u$ can be approximated to about 1 part in 40,000 if we choose the variables

$$x = 10000\,u, \qquad y = 20000 \log u.$$

A small portion of the log function approximation looks like this

Log Approximation

x	u	$\log u$	$20000 \log u$	y	$\frac{y}{20000} - \log u$
.	.	.	.	.	.
.	.	.	.	.	.
12010	1.2010	.079543	1590.86	1591	+.000007
12011	1.2011	.079579	1591.58	1592	+.000021
12012	1.2012	.079615	1592.30	1592	−.000015
12013	1.2013	.079651	1593.02	1593	−.000001
12014	1.2014	.079687	1593.74	1594	+.000013
12015	1.2015	.079723	1594.46	1594	−.000023
.	.	.	.	.	.
.	.	.	.	.	.

There are a number of ways of mechanizing products. For positive numbers only, we can use logarithms, in the form

$$t = \ln u, \qquad w = \ln v, \qquad z = e^{t+w}.$$

Another possible relation is trigonometric. Let U and V be constants that are larger than the largest values of u and v to be multiplied. Let

$$t = \arcsin (u/U), \qquad w = \arcsin (v/V),$$

then

$$\begin{aligned} uv &= U \sin t \cdot V \sin w, \\ &= \tfrac{1}{2}UV [\cos (t - w) - \cos (t + w)]. \end{aligned}$$

This is in a form that can be represented by a counting computer. A slight modification would give us such expressions as $r \sin u$, $r \cos u$, etc.

Arbitrary functions of several variables can be approximated by trigonometric sums. If the number of terms in such sums is not too great, the functions can be mechanized by function units. As a simple case, consider a function of the two variables, u and v, defined for u and v between $-\pi$ and π. Under favorable conditions the function can be approximated by a few terms of the trigonometric series

$$F(u, v) = \sum_j \sum_k F_{jk} e^{ijv+iku}$$
$$= \sum_j \sum_k F'_{jk} \cos (ju + kv) + \sum_j \sum_k F''_{jk} \sin (ju + kv).$$

Each argument, $ju + kv$, is obtainable by the use of gears and adders, each sine and cosine function can be represented by a function unit, the constants F'_{jk} and F''_{jk} become gear trains and $F(u, v)$ is, finally, obtained by adders that combine the outputs of the function units.

We return to the consideration of the properties of the class of counting computers as they are related to other computers.

In the point of accuracy, digital computers are practically unlimited. Capacities of 10^{10}, 10^{24} and even 10^{45} have been attained. The cost, bulk and slowness of operation of digital computers are roughly proportional to the number of digits, i.e. to the logarithms of the capacity. Counting computers are limited only by the bulk of the record and by the time required to pass from one part of the record to another. The bulk and slowness are roughly proportional to the capacity of the computer. Capacities of 10,000 to 50,000 distinct values are obtainable in practice with counting computers. Continuous computers are inherently relatively low-precision devices by comparison, being limited by the errors in measurement. Their cost and capacity are not related by any simple rule, but beyond a certain point, very great increase in cost produces only a slight increase in capacity. To hold errors below 1 part in 10,000 is almost impossible.

Digital computers are either restricted to simple operations, like multiplication and division, or else are extremely elaborate general-purpose machines. If we include the human operator as part of the system, it may be said that digital computing systems are complicated and flexible. Counting computers are less flexible, because they are characteristically capable of solving problems with special functional relations. Continuous computers are usually extremely special, although there are exceptions, like the slide rule. This specialization on the part of counting and continuous computers is not logically necessary; it comes about because these devices are relatively cheap, and can therefore be applied to solve special problems at a reasonable cost. To build a digital computer for a special problem would be prohibitively expensive. Furthermore both continuous and counting computers can be built with a great variety of functional relations.

It is generally recognized that the automatic control of machine tools and other devices is closely related to automatic computation, and in fact, that machine control is essentially a computing process that may, in some instances, be highly involved and demand great accuracy. Because counting

computers are essentially more precise than continuous computers they are well adapted to such controls.

In view of the foregoing discussion, we conclude that counting computers form a new category, being neither digital nor continuous. This class is intermediate between digital and continuous ones in capacity and in the generality of problems handled by one computer. Finally, because counting computers are capable of higher precision than are continuous devices they are adapted to controlling the production of the latter, this control being a kind of computation in a broad sense.

G. R. STIBITZ

EDITORIAL NOTE: On May 16, 1948 Dr. Stibitz issued a somewhat elaborate Report no. 1309. *Function Unit Theory of Computing Counters.* 120 mimeographed p. 21 × 28 cm. This summarizes and extends the work done in recent reports. Copies of this Report may be obtained from manufacturers of the Function Unit Computers, Taller & Cooper Inc., 75 Front St., Brooklyn, N. Y.

BIBLIOGRAPHY Z–IV

1. WILLIS A. ADCOCK, "An automatic simultaneous equation computer and its use in solving secular equations," *Rev. Sci. Instrs.*, v. 19, March 1948, p. 181–187.

Synopsis: "A feed-back method is described which is suitable for an automatic computer solving linear simultaneous equations and secular equations. This method is more general than is the Gauss-Seidel method. The design for an analog computer based on this feed-back method is outlined. This design utilizes resistive voltage dividers to represent coefficients and voltages to represent the variables. The variables are automatically adjusted by the feed-back system. The performance obtained on a 4-equation model is given. Usually an accuracy of better than one percent is obtained."

2. I. Îa. AKUSHSKIĬ, (a) "The four-counter scheme of solution of Dirichlet's problem by means of punched-card machines"; (b) "On numerical solution of Dirichlet problem on punched-card machines," Akad. N., SSSR (*Dok.*), C.R., v. 54, 1946, (a) p. 659–662; (b) p. 755–758. Compare *MTAC*, v. 3, p. 60. See *Math. Rev.*, v. 9, 1948, p. 105, P. W. KETCHUM.

3. I. Îa. AKUSHSKIĬ, "Protsess diagonal'nogo summirovaniîa na tabulîatore i nekotorye ego primeneniîa" [The process of diagonal summation on a tabulator and some of its applications], Akad. N., SSSR, *Izvestiîa, Otdelenie tekhnicheskikh N.*, 1947, p. 475–495. See *Math. Rev.*, v. 9, 1948, p. 105, D. H. L.

4. ANON., "Shop shots at Friden," with illustrations, *Amer. Machinist*, v. 92, Feb. 12, 1948, p. 98–99.

5. ANON., "The slide rule," *Life Insurance Courant*, v. 52, 1947, p. 370–372, 412–414.

6. P. V. H. WEEMS, "Navigation computer series," *Navigation*, v. 1, June, 1947, p. 134–137.

A list of more than 70 aerial and marine mechanical computers, many of the straight and circular slide rule type, being assembled for exhibition in the Hayden Planetarium.

7. G. I. GAVRILKO, "Noviĭ konikograf" [A new conicograph], *Geometrichniĭ Zbirnik*, Kharkov Univ., Ukrainskiĭ Institut Matematiki i Mekhaniki, Kharkov, v. 2, 1940, p. 107–108.

An instrument consisting of jointed links and slides.

8. A. LEĬBIN, "Polîarograf-Prilad dlîa pobudovi vzaemno polîarnikh krivikh" [Polarigraph-apparatus for drawing reciprocally polar curves], *Geometrichniĭ Zbirnik*, v. 2, 1940, p. 109–114.

A description of an instrument invented by the author, consisting of jointed links and slides, for drawing, with respect to a given circle, the polar of any given curve.

9. D. C. DEPACKH, "A resistor network for the approximate solution of the Laplace equation" *Rev. Sci. Instrs.*, v. 18, 1947, p. 798–799. See *Math. Rev.*, v. 9, 1948, p. 160, G. KRON.

NOTES

93. A FILM OF PART OF KULIK'S MAGNUS CANON FOR SALE.—In *MTAC*, v. 2, p. 139–140, some details were given concerning this great 8-volume factor table which became the property of the Academy of Sciences, Vienna. Professor D. N. LEHMER secured a photostatic copy of the latter part of volume 1 of this table, that is, for the numbers 9 000 000 to 12 642 600 inclusive. Of this photostatic copy the Carnegie Institution of Washington (1530 P Street, N.W.) has made (in 1947) a negative microfilm. The Institution is prepared to supply positive microfilm copies at $1.00 per film.

94. LAMBERTIAN OR LAMBDA FUNCTION.—Let the circle (1) $x^2 + y^2 = 1$ and the hyperbola (2) $x^2 - y^2 = 1$ with common center C be tangent at Q, denoting the common tangent there by l. Let q be any point on (2), and u be the area of the hyperbolic sector qCQ. Project q on l at P and let the angle $PCQ = \omega$. In his memoir "Observations trigonométriques," *Histoire de l'Académie Royale des Sciences*, Berlin, for the year 1768, 1770, p. 327–354, J. H. LAMBERT gave the formula

$$u = \ln \tan (45° + \tfrac{1}{2}\omega)$$

and also (p. 353–354) a table which professedly gives the value of u, to 7D, for $\omega = 0(1°)90°$. What Lambert really gives, however, is the values of $\log \tan (45° + \tfrac{1}{2}\omega)$ so that in order to get the corresponding values for u all the approximate values of the table must be multiplied by 2.30258 509. Lambert's table is given also in his *Zusätze zu den logarithmischen und trigonometrischen Tabellen*, Berlin, 1770, p. 176–181; in the FELKEL edition of this, Lisbon, 1798, p. 164–168; and in Lambert, *Opera Mathematica*, v. 2, 1948 (see RMT 521). [J.F.W. GRONAU, *Tafeln für die hyperbolischen Sectoren* (also as Neueste Schriften der naturf. Gesell. in Danzig, v. 6, Heft 4), Danzig, 1862, gives a table of $\log \tan (45° + \tfrac{1}{2}\omega)$, for $\omega = [10'(10')5°(1')83°(10')90°$; 5 or more D].

$$u = \int_0^\omega \sec x dx = \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\omega) = \ln (\sec \omega + \tan \omega) = gd^{-1}\omega = \operatorname{lam} \omega,$$

according to a notation suggested by Professor E. V. HUNTINGTON,[1] to connect the function in name with Lambert. $\omega = \text{lam}^{-1} u = gd\, u =$ the gudermannian of u, a name introduced by CAYLEY (1862).

With a possible exception to be described later, the first tables of u were published by LEGENDRE (a) T.4, $\omega = [0(0^\circ.5)90^\circ;$ 12D], Δ^5, (b) T.9, $\omega = [0(1^\circ)90^\circ;$ 9D], both of them in *Exercices de Calcul Intégral*, Paris, 1816, p. 160–163, 414–416; and also in *Traité des Fonctions Elliptiques*, v. 2, Paris, 1826, p. 256–259, 361–363.

Other tables of lam ω with the argument in degrees or parts of a degree are those of SAKAMOTO,[2] $\omega = [0(1''$ to $30'')1^\circ 2'30'';$ 9D], Δ, $[1^\circ(1'$ to $4')88^\circ;$ 6D], Δ, $[88^\circ(2''$ to $1')90^\circ;$ 6D], Δ^3 (radian arguments also given throughout); CAYLEY,[3] HOBSON,[4] and BOLL,[5] $\omega = [0(1^\circ)90^\circ;$ 7D] (radian arg. also given); POTIN,[6] and VASSAL,[7] $\omega = [0(1')90^\circ;$ 5D]; FORTI,[8] $\omega = [0(2')90^\circ;$ 5D]; HAYASHI,[9] $\omega = [0(10')90^\circ;$ 5D]; DALE,[10] $\omega = [0(30')90^\circ;$ 5D], Δ; and GREENHILL,[11] $\omega = [0(1^\circ)90^\circ;$ 5D] (radian arg. also given).

Two other tables of lam ω with argument in radians are those of MILNE-THOMSON & COMRIE,[12] $\omega = [0(.01)1;$ 4D], Δ, $[1(.01)1.47(.001)1.57;$ 3D], Δ; and of HALL,[13] $\omega = [0(.01)1.57;$ 4D].

The second published table of lam ω was that of GUDERMANN [14] with main argument in grades, (a) $\omega = [0(0^g.01)100^g;$ 7D], Δ (sexagesimal arguments also given); (b) $[88^g(0^g.01)100^g;$ 11D]; in the same argument were tables by POTIN,[15] $\omega = [0(0^g.01)100^g;$ 5D]; and by HOÜEL,[16] $\omega = [0(0^g.1)95^g;$ 4D], $[95^g(0^g.1)100^g;$ 3D], Δ.

Gudermann, a favorite teacher of WEIERSTRASS and a notable contributor to hyperbolic functions, used k for ω and $\pounds k$ for u, and his elaborate table to every centesimal minute is for $u = \pounds k$. This table is entitled "Tabelle der Längezahlen . . . aller Kreisbogen für den Radius = 1 . . . behufs der Zurückführung der hyperbolischen Functionen auf die cyklischen und umgekehrt." In Gudermann's notation $\pounds k$, the $\pounds$ and k were chosen because "Länge" and "Kreisbogen" are involved.

In the United States, at least, one meets also with the term "lambda function," $u = \lambda(\omega) = \text{lam}\,\omega$; for example, by G. H. CHANDLER [17] in 1907. Was the choice made because "Länge" was regarded as fundamental? Or was "Legendre" or "Lambert" in the thought of the originator? The term is also used in H. B. DWIGHT, *Tables of Integrals and Other Mathematical Data*, New York, 1934, and rev. ed., 1947.

There are also a number of tables of lam ω expressed in minutes of arc, namely: $(1/\pi)10800'$ lam ω, when ω is given in degrees and minutes. A list of some of these is given in FMR, *Index*, p. 185; for example, there is a table to the nearest $0'.01$, $\omega = 0(1')90^\circ$ in BECKER & VAN ORSTRAND, *Hyperbolic Functions*, fifth reprint, 1942, p. 309–318, which was taken from some edition of J. INMAN, *Nautical Tables*, say, London, 1858, p. 364–372. See MTE 129. Such tables of "meridional parts" for a spherical earth give the distances of the parallel of latitude ω from the equator on a MERCATOR (1512–1594) chart, in terms of the chart distance representing $1'$ of longitude at the equator.

The first table of this kind was the very celebrated one given by EDWARD WRIGHT (1558?–1615) in his *Certaine Errors in Navigation, Arising either of the ordinarie erroneous making or vsing of the sea Chart, Compasse, Crossestaffe, and Tables of declination of the Sunne, and fixed Starres detected and*

corrected. By E. W. London, 1599, 279 p.; there were two different t.p. imprints in this year. "A Table for the true diuiding" occurs on p. [43–48]; this is a 3–6S table throughout the quadrant interval 10′. In the third edition, 1657, by Joseph Moxon, which I have also inspected in the John Carter Brown Library of Brown University, this table has been expanded, p. 14–36, at interval 1′. Through the gracious kindness of Mr. HENRY TAYLOR of New York, I have also inspected a copy of the second very rare edition, "with many additions," 1610, with this same expanded "table of latitudes." A few extracts from the third table page of the 1599 edition are given by F. CAJORI, on p. 97 of *Napier Tercentenary Memorial Volume*, ed. C. G. KNOTT, London, 1915.

Thus Wright finally computed his table by the continued addition of the secants of 1′, 2′, 3′, etc. (Cajori errs, p. 96, in stating 1″, 2″, 3″) and thereby secured an approximation sufficiently exact for the mariner's use. HENRY BOND, a seventeenth century teacher of navigation (see C. HUTTON, *Philos. and Math. Dict.*, v. 1, 1815), discovered by chance that Wright's table was analogous to a scale of logarithmic tangents of half the complement of latitude. He published his discovery in an *Addition* to RICHARD NORWOOD's *Epitome*, 1645. JAMES GREGORY (1638–1675) was the first to demonstrate that Bond's observation was correct, in showing that $\int \sec x\,dx = \ln(\sec x + \tan x)$, in his *Exercitationes Geometricae*, London, 1668. See *James Gregory Tercentenary Memorial Volume* . . ., ed. by H. W. TURNBULL, London, 1939, p. 18, 236, 459–464. See also J. Gregory, "Analogia inter lineam meridianam planispherii nautici, et tangentes artificiales, geometrice demonstrata, &c.," F. MASERES, *Scriptores Logarithmici*, v. 2, London, 1791, p. 6f.

About a century after Wright's table appeared, EDMOND HALLEY (1656–1742) wrote an interesting paper [18] summarizing the work of prominent mathematicians proving Bond's surmise, and offering his own method for constructing the table. In particular he notes that "The Last, or 89°59′ is 30374.9634311414228643, and not 32348.5279 as Mr. *Wright* has it, by the addition of the *Secants* of every whole Minute."

A biography of Edward Wright was one of the 904 such sketches written for the *Dict. Nat. Biog.* (v. 63, 1900) by J. K. LAUGHTON, whose output in this case is not without blemish. One of his errors would have been avoided if he had consulted W. W. R. BALL, *A History of the Study of Mathematics at Cambridge*, Cambridge, 1889, p. 25–27. A. DEMORGAN's excellent sketch in *The Penny Cyclopaedia*, v. 27, London, 1843, may also be noted. On Mercator's chart of 1556 the lengthening of the degrees of latitude had no foundation in scientific theory. In this connection Wright's contribution based on solid mathematical foundations, before the integral calculus had been developed, was a notable achievement.

In 1599 Wright published also *The Haven finding Art, or, the way to find any Haven or place at Sea, by the latitude and variation.* This was an adaptation and extension of SIMON STEVIN, *De Havenvinding*, Leyden, 1599, of which there were in 1599 also French and Latin editions. The 1657 reprint (20 p.) of Wright's edition, appended to the third edition of his *Certaine Errors*, is overlooked by E. J. DIJKSTERHUIS in his *Simon Stevin*, The Hague, 1943; and also by BIERENS DE HAAN, *Bibliographie Néerlandaise* . . . Rome, 1883. There was then no way of determining the longitude at sea. Wright effected a revolution in the science of navigation. Through Mr.

Taylor's courtesy Brown University filmed for its Library both the French edition of Stevin's *De Havenvinding* and Wright's *Certaine Errors*, 1610.

When the invention of logarithms became public Wright applied himself to the study of the new method and translated Napier's Description of his Canon. This translation was forwarded to Napier at Edinburgh, received his approbation and a few lines of addition, and was returned for publication. But Wright died soon after receiving it back in 1615. It was published in 1616 by his son SAMUEL WRIGHT. The so-called 1618 edition is simply that of 1616 with the title page cut out and a new one substituted.

R. C. A.

[1] A. E. KENNELLY, "Gudermannians and Lambertians with their respective addition theorems," Amer. Phil. Soc., *Proc.*, v. 68, 1929, p. 179. The well-known formula given here on p. 183, namely: lam $2\omega = 2 \tanh^{-1} (\tan \omega)$, was intended in the 1909 edition of G. F. BECKER & C. E. VAN ORSTRAND, *Smithsonian Mathematical Tables. Hyperbolic Functions*, with $gd^{-1}\, 2\omega$ for lam 2ω, p. xv (see *MTAC*, v. 2, p. 311).

[2] SAMATA SAKAMOTO, *Tables of Gudermannian Angles and Hyperbolic Functions*, Tokyo, 1934, p. 14–94.

[3] A. CAYLEY, "On the orthomorphosis of the circle into the parabola," *Quart. Jn. Math.*, v. 20, 1885, p. 220; also in *Coll. Math. Papers*, v. 12, 1897, p. 336.

[4] E. W. HOBSON, *A Treatise on Plane Trigonometry*. Cambridge, 1891, and second ed., 1897, p. 316; third ed., 1911, fourth ed., 1918, and fifth ed., 1921, p. 336.

[5] M. BOLL, *Tables Numériques Universelles*, Paris, 1947, p. 487.

[6] L. POTIN, *Formules et Tables Numériques* . . . Paris, 1925, p. 450–494.

[7] V. VASSAL, *Nouvelles Tables donnant avec cinq Décimales* . . . Paris, 1872, p. [67]–[111].

[8] A. FORTI & O. F. MOSSOTTI, "Tavole dei logaritmi delle funzioni circolari ed iperboliche," filling the whole of *Annali delle Università Toscane*, Pisa, v. 6, 1863, 4to; Forti's part of the work consists of the tables on [228] unnumbered pages, and the introduction, p.27–48; table of u, p. [183–228]. This is preceded by Mossotti's "Teoria ed applicazioni delle funzioni circolari ed iperboliche," p. 7–26 + plate. In 1863 this work seems to have been published separately at Pisa with the following title: *Tavole dei Logaritmi delle Funzioni Circolari ed Iperboliche, precedute dalla Storia e Teoria delle Funzioni stesse e da Applicazioni*. Second ed., *Tavole di Logaritmi dei Numeri e delle Funzioni Circolari ed Iperboliche, precedute dalla Storia e Teoria delle Iperboliche, da Applicazioni, e da altre Tavole di Uso Frequente*. Turin, Florence, Milan, Paravia & Co., 2 v., 1870; third ed., Turin and Rome, 1877, 584 p.

[9] K. HAYASHI, *Fünfstellige Funktionentafeln* . . . Berlin, 1930, p. 2–19.

[10] J. B. DALE, *Five-Figure Tables of Mathematical Functions*, London, 1903, p. 67.

[11] G. GREENHILL, *The Applications of Elliptic Functions*, London, 1892, p. 16. French ed., Paris, 1895, p. 569.

[12] L. M. MILNE-THOMSON & L. J. COMRIE, *Standard Four-Figure Mathematical Tables*, London, 1931, p. 208.

[13] W. HALL, *Tables and Constants to Four Figures*, Cambridge, 1905, p. 48–49.

[14] C. GUDERMANN, "Potenzial- oder cyklisch-hyperbolische Functionen," *Jn. f. d. reine u. angew. Math.*, v. 7, 1831, p. 72–96, 176–200; v. 8, 1832, p. 64–116; v. 9, 1832, p. 362–378. Reprinted in *Theorie der Potenzial- oder cyklisch-hyperbolischen Functionen*, Berlin, 1833, p. 159–260, 337–350.

[15] L. POTIN, *Formules et Tables Numériques*, Paris, 1925, p. 496–595.

[16] The Société d. Sciences Physiques et Naturelles de Bordeaux, *Mémoires*, v. 4; Cahier 2, 1866, contains the first edition complete, lxxi, [64], 2 p. of J. HOÜEL, *Recueil de Formules et de Tables Numériques*, printed at Paris by Gauthier-Villars. On the title page of this first edition appears also "Extrait des Mémoires de la Soc. d. Sci. phys. et nat. de Bordeaux." The table in which we are interested occurs on p. [36]–[55]. Second ed., a reprint, Paris, 1868; third ed., 1885; third ed. reprinted, 1901; third ed. rev. and corrected, 1927.

[17] G. H. CHANDLER, *Elements of the Infinitesimal Calculus*. Third ed. rewritten, New York, 1907, p. 298.

[18] E. HALLEY, "An easie demonstration of the analogy of the logarithmick tangents to the meridian line or sum of the secants: with various methods for computing the same to the utmost exactness," R. Soc. London, *Phil. Trans.*, v. 19, no. 219, Jan.–Feb. 1695/6, p. 202–214; the v. is dated 1698. Also in his *Miscellanea Curiosa*, second ed., v. 2, 1708, p. 20–36; and third ed., v. 2, 1723, p. 20–36.

QUERIES

27. First Table of $\tan^{-1} x$.—When was the first table of $\tan^{-1} x$ published? In *MTAC*, v. 3, p. 42, it is suggested that this may have been in 1819 when the table of William Spence first appeared. The following copy of a letter addressed to J. W. L. Glaisher indicates that this date may be set back more than 50 years:

"20, Girdler's Road
Brook Green. W.
30 Sept. 1877

"My Dear Glaisher

My son has just brought me an old German schoolbook of 1764—Segner, Anfangsgründe der Arithmetik Geometrie u.s.w., Halle. It contains a curiously arranged table, the argument being the natural numbers from 1–1000, and the entries (1) the common logs; (2) $\sin^{-1}(x/1000)$; (3) $\tan^{-1}(x/1000)$; (4) $\cot^{-1}(x/1000)$. It is a very coarse expedient for contraction; but I never saw one before, and I thought you might like to hear of it in case it should be new to you also.

Yours ffly
C. W. Merrifield

The book is strictly an octavo of very small size, the page 7″ × 4″." [There are various references to Merrifield in *MTAC*, v. 1.]

The book here referred to was evidently written by Johann Andreas von Segner (1704–1777), professor of physics and mathematics at the University of Göttingen (1735–1755) and then professor in the same subjects at the University of Halle. A long list of his publications is given in "Poggendorff," v. 2, one entry being as follows: "Elementa arithmeticae et geometriae, 8°, Gotting. 1739, umgearbeitet unter d. Titel: Elementa arithmeticae geometriae et calculi geometrici, 8°, Halae 1756 et 1767 (deutsch von seinem Sohne Johann Wilhelm, 8°, Ib. 1764 u. 1773)." Thus the very book that Merrifield lists was a German translation, of one of Segner's Latin books, by his son. Where may a copy of this translation, or of the Latin original, be seen? The possibility of the $\tan^{-1} x$ table appearing in 1756, or even in 1739, is here definitely suggested. This Segner work of 1739 is mentioned in A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, v. 2, Leipzig, 1903, p. 89, but tabular details are not mentioned.

In v. 1, 1900, p. 84–86, of Braunmühl's work is material about Abû'l Hasan Ali, of Morocco, a thirteenth century scholar whose notable Arabic compilation was translated into French and published by J. J. Sédillot as *Traité des Instruments Astronomiques des Arabes*, 2 v., Paris, 1834. In v. 1, p. 170, there is a table equivalent to $y = \tan^{-1} x$, where $y = 1°(1°)60°$, and $60x$ equals the number of parts, minutes and seconds, given in the table, from $1^p2'51''$ to $103^p55'22''$. The gnomon ($=60$) is here horizontal and the shadow vertical. Thus, while this table is not the earliest published, it seems to be the most ancient table of $\tan^{-1} x$ now known. Compare *MTAC*, v. 2, p. 21.

There are, however, much older tables of $\cot^{-1} x$. On p. 168–169 of the same Sédillot volume is a table equivalent to $y = \cot^{-1} x$, where the horizontal shadow varies from 1(1)140 and the gnomon = 12, the corresponding

angles being given in degrees and minutes. This same table was given much earlier by AL-KHOWĀRIZMĪ (fl. about 825); see H. SUTER, *Die astronomischen Tafeln des Muhammed Ibn Mūsā Al-Khowārizmī*, Copenhagen, 1914, Table 60, p. 174 (Danske Vidensk. Selsk., *Skrifter*, 7s., *Hist. og Filos.*, v. 3, no. 1).

R. C. A.

QUERIES—REPLIES

35. TABLES OF $\tan^{-1}(m/n)$ (Q **14**, v. 1, p. 431; QR **18**, v. 1, p. 460; **20**, v. 2, p. 62; **24**, v. 2, p. 147; **28**, v. 2, p. 287).—A table has been prepared which expresses $\tan^{-1}(m/n)$, for $0 < m < n \leq 50$, $0 < m < n = 100$, as a sum of multiples of $\tan^{-1} n_i$ where the n_i are fundamental in the sense of QR**28**. From this table a table of $\tan^{-1}(m/n)$ can be obtained by addition of values from such tables as that of the NBSCL. The manuscript is in the possession of the National Bureau of Standards.

JOHN TODD

NBSINA

CORRIGENDA

V. 1, p. 59, l. −21, *delete* correct.

V. 2, p. 137, $F(\theta, \phi)$, at $\theta = \phi = 86°$, substitute: *for* 3.17204 1744, *read* 3.17030 9981[1].

V. 3, p. 84, l. −6, −5, *for* It may be the first six-place table of the kind, but as long ago, *read* As long ago; p. 106, l. 19–20, *for* .49, *read* −.49; p. 129, l. 23, *for* A. H. BURKS, *read* A. W. BURKS. These errors in v. 3 were due to errata in the texts which were being reviewed.

Messrs. B. L. COLEMAN & SIDNEY MICHELSON of The British Electrical and Allied Industries Research Association, 5 Wadsworth Road, Greenford, Middx., England, reported the following, which Professor LEHMER accepts: v. 1, p. 379, l. 20–21, *for* $B_2^{(n)} = \alpha_1^n \alpha_2^{n-1}$ *read* $B_2^{(n)} = \alpha_1^n \alpha_2^{n-1} + \alpha_1^{n-1} \alpha_2^n$; *for* $B_3^{(n)} = -\alpha_1^n \alpha_2^n \alpha_3^{n-1}$, *read* $B_3^{(n)} = -[\alpha_1^n \alpha_2^n \alpha_3^{n-1} + \alpha_1^n \alpha_2^{n-1} \alpha_3^n + \alpha_1^{n-1} \alpha_2^n \alpha_3^n]$. Equation (11) should then be

$$1/\alpha_1 = B_1^{(n)}/A_1^{(n)},$$
$$1/\alpha_2 = B_2^{(n)}/A_2^{(n)} - B_1^{(n)}/A_1^{(n)},$$
$$1/\alpha_3 = B_3^{(n)}/A_3^{(n)} - B_2^{(n)}/A_2^{(n)}.$$

In general, if the equation has real and complex roots, and α_k is simple, it is given by

$$1/\alpha_k = B_k^{(n)}/A_k^{(n)} - B_{k-1}^{(n)}/A_{k-1}^{(n)},$$

whilst complex α's are given by the expression (13) on page 380.